



ИТОГИ НАУКИ И ТЕХНИКИ.
Современная математика и ее приложения.
Тематические обзоры.
Том 217 (2022). С. 107–137
DOI: 10.36535/0233-6723-2022-217-107-137

УДК 512.7

ПОЛИНОМИАЛЬНЫЕ АВТОМОРФИЗМЫ, КВАНТОВАНИЕ
И ЗАДАЧИ ВОКРУГ ГИПОТЕЗЫ ЯКОБИАНА.
V. ГИПОТЕЗА ЯКОБИАНА И ПРОБЛЕМЫ
ТИПА ШПЕХТА И БЕРНСАЙДА

© 2022 г. А. М. ЕЛИШЕВ, А. Я. КАТЕЛЬ-БЕЛОВ,
Ф. РАЗАВИНИЯ, Ц.-Т. ЮЙ, В. ЧЖАН

Аннотация. Работа является завершающей частью обзора результатов, касающихся квантового подхода к некоторым классическим аспектам некоммутативных алгебр. Первая часть: Итоги науки и техники. Современная математика и ее приложения. Тематические обзоры. — 2022. — 213. — С. 110–144. Вторая часть: Итоги науки и техники. Современная математика и ее приложения. Тематические обзоры. — 2022. — 214. — С. 107–126. Третья часть: Итоги науки и техники. Современная математика и ее приложения. Тематические обзоры. — 2022. — 215. — С. 95–128. Четвертая часть: Итоги науки и техники. Современная математика и ее приложения. Тематические обзоры. — 2022. — 216. — С. 153–171.

Ключевые слова: автоморфизм, квантование, гипотеза о Якобиане.

POLYNOMIAL AUTOMORPHISMS, QUANTIZATION,
AND JACOBIAN CONJECTURE RELATED PROBLEMS.
V. JACOBIAN CONJECTURE
AND SPECHT AND BURNSIDE TYPE PROBLEMS

© 2022 A. M. ELISHEV, A. Ya. KANEL-BELOV,
F. RAZAVINIA, J.-T. YU, W. ZHANG

ABSTRACT. This paper is the final part of a review of results concerning the quantization approach to the some classical aspects of noncommutative algebras. The first part is: Itogi Nauki Tekhn. Sovr. Mat. Prilozh. Temat. Obzory, **213** (2022), pp. 110–144. The second part is: Itogi Nauki Tekhn. Sovr. Mat. Prilozh. Temat. Obzory, **214** (2022), pp. 107–126. The third part is: Itogi Nauki Tekhn. Sovr. Mat. Prilozh. Temat. Obzory, **215** (2022), pp. 95–128. The fourth part is: Itogi Nauki Tekhn. Sovr. Mat. Prilozh. Temat. Obzory, **216** (2022), pp. 153–171.

Keywords and phrases: automorphism, quantization, Jacobian conjecture.

AMS Subject Classification: 14R10, 18G85

CONTENTS

Chapter 5. Jacobian conjecture and Specht and Burnside type problems	108
5.1. The Jacobian Conjecture and Burnside type problems, via algebras	110
5.2. The Jacobian Conjecture for varieties, and deformations	118
Conclusion	129
References	129

CHAPTER 5

JACOBIAN CONJECTURE AND SPECHT AND BURNSIDE TYPE PROBLEMS

This chapter explores an approach to polynomial mappings and the Jacobian Conjecture and related questions, initiated by A. V. Yagzhev, whereby these questions are translated to identities of algebras, leading to a solution in [217] of the version of the Jacobian Conjecture for free associative algebras. (The first version, for two generators, was obtained by Dicks and Levin (see [73, 74]), and the full version by Schofield [175].) We start by laying out the basic framework in this introduction. Next, we set up Yagzhev's correspondence to algebras in In Sec. 5.1, leading to the basic notions of weak nilpotence and Engel type. In Sec. 5.2 we discuss the Jacobian Conjecture in the context of various varieties, including the free associative algebra.

Given any polynomial endomorphism ϕ of the n -dimensional affine space $A_{\mathbf{k}}^n = \text{Spec } \mathbf{k}[x_1, \dots, x_n]$ over a field \mathbf{k} , we define its *Jacobian matrix* as the matrix $(\partial\phi^*(x_i)/\partial x_j)_{1 \leq i, j \leq n}$. The determinant of the Jacobian matrix is called the *Jacobian* of ϕ . The celebrated *Jacobian Conjecture* JC_n in dimension $n \geq 1$ asserts that *for any field \mathbf{k} of characteristic zero, any polynomial endomorphism ϕ of $A_{\mathbf{k}}^n$ having Jacobian 1 is an automorphism*. Equivalently, one can say that ϕ preserves the standard top-degree differential form $dx_1 \wedge \dots \wedge dx_n \in \Omega^n(A_{\mathbf{k}}^n)$. References to this well known problem and related questions can be found in [21, 135, 202]. By the Lefschetz principle it is sufficient to consider the case $\mathbf{k} = \mathbb{C}$; obviously, JC_n implies JC_m if $n > m$. The conjecture JC_n is obviously true in the case $n = 1$, and it is open for $n \geq 2$.

The *Jacobian Conjecture*, denoted as JC , is the conjunction of the conjectures JC_n for all finite n . The Jacobian Conjecture has many reformulations (such as the Kernel Conjecture and the Image Conjecture, cf. [90, 93, 202, 226, 227] for details) and is closely related to questions concerning quantization. It is stably equivalent to the following conjecture of Dixmier, concerning automorphisms of the Weyl algebra W_n , otherwise known as the *quantum affine algebra*.

Dixmier Conjecture DC_n . *Is $\text{End}(W_n) = \text{Aut}(W_n)$?*

The implication $DC_n \rightarrow JC_n$ is well known, and the inverse implication $JC_{2n} \rightarrow DC_n$ was recently obtained independently by Tsuchimoto [191] (using p -curvature) and Belov and Kontsevich [41, 42] (using Poisson brackets on the center of the Weyl algebra). Bavula [30] has obtained a shorter proof, and also obtained a positive solution of an analog of the Dixmier Conjecture for integro differential operators, cf. [28]. He also proved that every monomorphism of the Lie algebra of triangular polynomial derivations is an automorphism [29] (an analog of Dixmier's conjecture).

The Jacobian Conjecture is closely related to many questions of affine algebraic geometry concerning affine space, such as the Cancellation Conjecture (see Sec. 5.2.4). If we replace the variety of commutative associative algebras (and the accompanying affine spaces) by an arbitrary algebraic

variety¹, one easily gets a counterexample to the JC. So, strategically these questions deal with some specific properties of affine space which we do not yet understand, and for which we do not have the appropriate formulation apart from these very difficult questions.

It seems that these properties do indicate some sort of quantization. From that perspective, non-commutative analogs of these problems (in particular, the Jacobian Conjecture and the analog of the Cancellation Conjecture) become interesting for free associative algebras, and more generally, for arbitrary varieties of algebras.

We work in the language of universal algebra, in which an algebra is defined in terms of a set of operators, called its *signature*. This approach enhances the investigation of the Yagzhev correspondence between endomorphisms and algebras. We work with deformations and so-called *packing properties* to be introduced in Secs. 5.2 and 5.2.2.1, which denote specific noncommutative phenomena which enable one to solve the JC for the free associative algebra.

From the viewpoint of universal algebra, the Jacobian conjecture becomes a problem of “Burnside type,” by which we mean the question of whether a given finitely generated algebraic structure satisfying given periodicity conditions is necessarily finite, cf. [225]. Burnside originally posed the question of the finiteness of a finitely generated group satisfying the identity $x^n = 1$. (For odd $n \geq 661$, counterexamples were found by Novikov and Adian, and quite recently Adian reduced the estimate from 661 to 101). Another class of counterexamples was discovered by Ol’shanskij [151]. Kurosh posed the question of local finiteness of algebras whose elements are algebraic over the base field. For algebraicity of bounded degree, the question has a positive solution, but otherwise there are the Golod-Shafarevich counterexamples.

Burnside type problems play an important role in algebra. Their solution in the associative case is closely tied to Specht’s problem of whether any set of polynomial identities can be deduced from a finite subset. The JC can be formulated in the context of whether one system of identities implies another, which also relates to Specht’s problem.

In the Lie algebra case there is a similar notion. An element $x \in L$ is called *Engel of degree n* if $[\dots[[y, x], x] \dots, x] = 0$ for any y in the Lie algebra L . Zelmanov’s result that any finitely generated Lie algebra of bounded Engel degree is nilpotent yielded his solution of the Restricted Burnside Problem for groups. Yagzhev introduced the notion of *Engelian* and *weakly nilpotent* algebras of arbitrary signature (see Definitions 5.1.7 and 5.1.5), and proved that the JC is equivalent to the question of weak nilpotence of algebras of Engel type satisfying a system of Capelli identities, thereby showing the relation of the JC with problems of Burnside type.

A negative approach. Let us mention a way of constructing counterexamples. This approach, developed by Gizatullin, Kulikov, Shafarevich, Vitushkin, and others, is related to decomposing polynomial mappings into the composition of σ -processes [?, 96, 135, 177, 204–206]. It allows one to solve some polynomial automorphism problems, including tameness problems, the most famous of which is *Nagata’s Problem* concerning the wildness of Nagata’s automorphism

$$(x, y, z) \mapsto (x - 2(xz + y^2)y - (xz + y^2)^2z, y + (xz + y^2)z, z),$$

cf. [148]. Its solution by Shostakov and Umirbaev [183] is the major advance in this area in the last decade. The Nagata automorphism can be constructed as a product of automorphisms of $K(z)[x, y]$, some of them having non-polynomial coefficients (in $K(z)$). The following theorem of Abhyankar–Moh–Suzuki [2, 140, 189] can be viewed in this context:

AMS Theorem. *If f and g are polynomials in $K[z]$ of degrees n and m for which $K[f, g] = K[z]$, then n divides m or m divides n .*

Degree estimate theorems are polynomial analogs to Liouville’s approximation theorem in algebraic number theory ([49, 106, 137, 143]). T. Kishimoto has proposed using a program of Sarkisov, in particular for Nagata’s Problem. Although difficulties remain in applying “ σ -processes” (decomposition of birational mappings into standard blow-up operations) to the affine case, these may provide new

¹Algebraic geometers use word *variety*, roughly speaking, for objects whose local structure is obtained from the solution of system of algebraic equations. In the framework of universal algebra, this notion is used for subcategories of algebras defined by a given set of identities. A deep analog of these notions is given in [34].

insight. If we consider affine transformations of the plane, we have relatively simple singularities at infinity, although for bigger dimensions they can be more complicated. Blow-ups provide some understanding of birational mappings with singularities. Relevant information may be provided in the affine case. The paper [47] contains some deep considerations about singularities.

5.1. THE JACOBIAN CONJECTURE AND BURNSIDE TYPE PROBLEMS, VIA ALGEBRAS

In this section we translate the Jacobian Conjecture to the language of algebras and their identities. This can be done at two levels: At the level of the algebra obtained from a polynomial mapping, leading to the notion of *weak nilpotence* and *Yagzhev algebras* and at the level of the differential and the algebra arising from the Jacobian, leading to the notion of *Engel type*. The Jacobian Conjecture is the link between these two notions.

5.1.1. The Yagzhev correspondence.

5.1.1.1. Polynomial mappings in universal algebra. Yagzhev's approach is to pass from algebraic geometry to universal algebra. Accordingly, we work in the framework of a universal algebra A having signature Ω . $A^{(m)}$ denotes $A \times \cdots \times A$, taken m times.

We fix a commutative, associative base ring C , and consider C -modules equipped with extra operators $A^{(m)} \rightarrow A$, which we call *m-ary*. Often one of these operators will be (binary) multiplication. These operators will be multilinear, i.e., linear with respect to each argument. Thus, we can define the *degree* of an operator to be its number of arguments. We say an operator $\Psi(x_1, \dots, x_m)$ is *symmetric* if $\Psi(x_1, \dots, x_m) = \Psi(x_{\pi(1)}, \dots, x_{\pi(m)})$ for all permutations π .

Definition 5.1.1. A *string* of operators is defined inductively. Any operator $\Psi(x_1, \dots, x_m)$ is a string of degree m , and if s_j are strings of degree d_j , then $\Psi(s_1, \dots, s_m)$ is a string of degree $\sum_{j=1}^m d_j$. A mapping

$$\alpha : A^{(m)} \rightarrow A$$

is called *polynomial* if it can be expressed as a sum of strings of operators of the algebra A . The *degree* of the mapping is the maximal length of these strings.

Example 5.1.2. Assume that an algebra A has two extra operators: a binary operator $\alpha(x, y)$ and a tertiary operator $\beta(x, y, z)$. The mapping $F : A \rightarrow A$ given by $x \rightarrow x + \alpha(x, x) + \beta(\alpha(x, x), x, x)$ is a polynomial mapping of A , having degree 4. Note that if A is finite dimensional as a vector space, not every polynomial mapping of A as an affine space is a polynomial mapping of A as an algebra.

5.1.1.2. Yagzhev's correspondence between polynomial mappings and algebras. Here we associate an algebraic structure to each polynomial map. Let V be an n -dimensional vector space over the field \mathbf{k} , and $F : V \rightarrow V$ be a polynomial mapping of degree m . Replacing F by the composite TF , where T is a translation such that $TF(0) = 0$, we may assume that $F(0) = 0$. Given a base $\{e_i\}_{i=1}^n$ of V , and for an element v of V written uniquely as the sum $\sum x_i e_i$, for $x_i \in \mathbf{k}$, the coefficients of e_i in $F(v)$ are (commutative) polynomials in the x_i . Then F can be written in the following form:

$$x_i \mapsto F_{0i}(\mathbf{x}) + F_{1i}(\mathbf{x}) + \cdots + F_{mi}(\mathbf{x})$$

where each $F_{\alpha i}(\mathbf{x})$ is a homogeneous form of degree α , i.e.,

$$F_{\alpha i}(\mathbf{x}) = \sum_{j_1 + \cdots + j_n = \alpha} \kappa_{j_1 \dots j_n} x_1^{j_1} \cdots x_n^{j_n},$$

with $F_{0i} = 0$ for all i , and

$$F_{1i}(\mathbf{x}) = \sum_{k=1}^n \mu_{ki} x_k.$$

We are interested in invertible mappings that have a nonsingular Jacobian matrix (μ_{ij}) . In particular, this matrix is nondegenerate at the origin. In this case $\det(\mu_{ij}) \neq 0$, and by composing F with an affine transformation we arrive at the situation for which $\mu_{ki} = \delta_{ki}$. Thus, the mapping F may be taken to have the following form:

$$x_i \rightarrow x_i - \sum_{k=2}^m F_{ki}. \quad (5.1.1)$$

Suppose we have a mapping as in (5.1.1). Then the Jacobi matrix can be written as $E - G_1 - \dots - G_{m-1}$ where G_i is an $n \times n$ matrix with entries which are homogeneous polynomials of degree i . If the Jacobian is 1, then it is invertible with inverse a polynomial matrix (of homogeneous degree at most $(n-1)(m-1)$, obtained via the adjoint matrix).

If we write the inverse as a formal power series, we compare the homogeneous components and get:

$$\sum_{j_i m_{j_i} = s} M_J = 0, \quad (5.1.2)$$

where M_J is the sum of products $a_{\alpha_1} a_{\alpha_q}$ in which the factor a_j occurs m_j times, and J denotes the multi-index (j_1, \dots, j_q) .

Yagzhev considered the cubic homogeneous mapping $\mathbf{x} \rightarrow \mathbf{x} + (\mathbf{x}, \mathbf{x}, \mathbf{x})$, whereby the Jacobian matrix becomes $E - G_3$. We return to this case in Remark 5.1.9. The slightly more general approach given here presents the Yagzhev correspondence more clearly and also provides tools for investigating deformations and packing properties (see Sec. 5.2.2.1). Thus, we consider not only the cubic case (i.e. when the mapping has the form

$$x_i \rightarrow x_i + P_i(x_1, \dots, x_n); \quad i = 1, \dots, n,$$

with P_i cubic homogeneous polynomials), but the more general situation of arbitrary degree.

For any ℓ , the set of (vector valued) forms $\{F_{\ell,i}\}_{i=1}^n$ can be interpreted as a homogeneous mapping $\Phi_\ell : V \rightarrow V$ of degree ℓ . When $\text{char}(\mathbf{k})$ does not divide ℓ , we take instead the polarization of this mapping, i.e. the multilinear symmetric mapping

$$\Psi_\ell : V^{\otimes \ell} \rightarrow V$$

such that

$$(F_{\ell,i}(x_1), \dots, F_{\ell,i}(x_n)) = \Psi_\ell(\mathbf{x}, \dots, \mathbf{x}) \cdot \ell!$$

Then Eq. (5.1.1) can be rewritten as follows:

$$\mathbf{x} \rightarrow \mathbf{x} - \sum_{\ell=2}^m \Psi_\ell(\mathbf{x}, \dots, \mathbf{x}). \quad (5.1.3)$$

We define the algebra $(A, \{\Psi_\ell\})$, where A is the vector space V and the Ψ_ℓ are viewed as operators $A^\ell \rightarrow A$.

Definition 5.1.3. The *Yagzhev correspondence* is the correspondence from the polynomial mapping (V, F) to the algebra $(A, \{\Psi_\ell\})$.

5.1.2. Translation of the invertibility condition to the language of identities. The next step is to bring in algebraic varieties, defined in terms of identities.

Definition 5.1.4. A *polynomial identity* (PI) of A is a polynomial mapping of A , all of whose values are identically zero.

The *algebraic variety* generated by an algebra A , denoted as $\text{Var}(A)$, is the class of all algebras satisfying the same PIs as A .

Now we come to a crucial idea of Yagzhev: *The invertibility of F and the invertibility of the Jacobian of F can be expressed via (5.1.2) in the language of polynomial identities.*

Namely, let

$$y = F(x) = x - \sum_{\ell=2}^m \Psi_\ell(x).$$

Then

$$F^{-1}(x) = \sum_t t(x), \quad (5.1.4)$$

where each t is a *term*, a formal expression in the mappings $\{\Psi_\ell\}_{\ell=2}^m$ and the symbol x . Note that the expressions $\Psi_2(x, \Psi_3(x, x, x))$ and $\Psi_2(\Psi_3(x, x, x), x)$ are different although they represent same element of the algebra. Denote by $|t|$ the number of occurrences of variables, including multiplicity, which are included in t .

The invertibility of F means that, for all $q \geq q_0$,

$$\sum_{|t|=q} t(a) = 0, \quad \forall a \in A. \quad (5.1.5)$$

Thus we have translated invertibility of the mapping F to the language of identities. (Yagzhev had an analogous formula, where the terms only involved Ψ_3 .)

Definition 5.1.5. An element $a \in A$ is called *nilpotent* of index $\leq n$ if

$$M(a, a, \dots, a) = 0$$

for each monomial $M(x_1, x_2, \dots)$ of degree $\geq n$. A is *weakly nilpotent* if each element of A is nilpotent. A is *weakly nilpotent of class k* if each element of A is nilpotent of index k . (Some authors use the terminology *index* instead of *class*.) Equation (5.1.5) means A is weakly nilpotent.

To stress this fundamental notion of Yagzhev, we define a *Yagzhev algebra* of order q_0 to be a weakly nilpotent algebra, i.e., satisfying the identities (5.1.5), also called the *system of Yagzhev identities* arising from F .

Summarizing, we get the following fundamental translation from conditions on the endomorphism F to identities of algebras.

Theorem 5.1.6. *The endomorphism F is invertible if and only if the corresponding algebra is a Yagzhev algebra of high enough order.*

5.1.2.1. Algebras of Engel type. The analogous procedure can be carried out for the differential mapping. We recall that Ψ_ℓ is a symmetric multilinear mapping of degree ℓ . We denote the mapping $y \rightarrow \Psi_\ell(y, x, \dots, x)$ as $\text{Ad}_{\ell-1}(x)$.

Definition 5.1.7. An algebra A is of *Engel type s* if it satisfies a system of identities

$$\sum_{\ell m_\ell = s} \sum_{\alpha_1 + \dots + \alpha_q = m_\ell} \text{Ad}_{\alpha_1}(x) \cdots \text{Ad}_{\alpha_q}(x) = 0. \quad (5.1.6)$$

A is of *Engel type* if A has Engel type s for some s .

Theorem 5.1.8. *The endomorphism F has Jacobian 1 if and only if the corresponding algebra has Engel type s for some s .*

Proof. Let $x' = x + dx$. Then

$$\Psi_\ell(x') = \Psi_\ell(x) + \ell \Psi_\ell(dx, x, \dots, x) + \text{forms containing more than one occurrence of } dx. \quad (5.1.7)$$

Hence the differential of the mapping

$$F : \mathbf{x} \mapsto \mathbf{x} - \sum_{\ell=2}^m \Psi_\ell(\mathbf{x}, \dots, \mathbf{x})$$

is

$$\left(E - \sum_{\ell=2}^m \ell \text{Ad}_{\ell-1}(x) \right) \cdot dx$$

The identities (5.1.2) are equivalent to the system of identities (5.1.6) in the signature $\Omega = (\Psi_2, \dots, \Psi_m)$, taking $a_{\alpha_j} = \text{Ad}_{\alpha_j}$ and $m_j = \deg \Psi_\ell - 1$. \square

Thus, we have reformulated the condition of invertibility of the Jacobian in the language of identities.

As explained in [202], it is well known from [21, 220] that the Jacobian Conjecture can be reduced to the cubic homogeneous case; i.e., it is enough to consider mappings of type

$$x \rightarrow x + \Psi_3(x, x, x).$$

In this case the Jacobian assumption is equivalent to the *Engel condition* – nilpotence of the mapping $\text{Ad}_3(x)[y]$ (i.e. the mapping $y \rightarrow (y, x, x)$). Invertibility, considered in [21], is equivalent to weak nilpotence, i.e., to the identity $\sum_{|t|=k} t = 0$ holding for all sufficiently large k .

Remark 5.1.9. In the cubic homogeneous case, $j = 1$, $\alpha_j = 2$ and $m_j = s$, and we define the linear map

$$\text{Ad}_{xx} : y \rightarrow (x, x, y)$$

and the index set $T_j \subset \{1, \dots, q\}$ such that $i \in T_j$ if and only if $\alpha_i = j$.

Then Eq. (5.1.6) has the following form:

$$\text{Ad}_{xx}^{s/2} = 0.$$

Thus, for a ternary symmetric algebra, Engel type means that the operators Ad_{xx} for all x are nilpotent. In other words, the mapping

$$\text{Ad}_3(x) : y \rightarrow (x, x, y)$$

is nilpotent. Yagzhev called this the *Engel condition*. (For Lie algebras the nilpotence of the operator $\text{Ad}_x : y \rightarrow (x, y)$ is the usual Engel condition. Here we have a generalization for arbitrary signature.)

Here are Yagzhev's original definitions, for edification. A binary algebra A is *Engelian* if for any element $a \in A$ the subalgebra $\langle R_a, L_a \rangle$ of vector space endomorphisms of A generated by the left multiplication operator L_a and the right multiplication operator R_a is nilpotent, and *weakly Engelian* if for any element $a \in A$ the operator $R_a + L_a$ is nilpotent.

This leads us to the *Generalized Jacobian Conjecture*:

Conjecture. Let A be an algebra with symmetric \mathbf{k} -linear operators Ψ_ℓ , for $\ell = 1, \dots, m$. In any variety of Engel type, A is a Yagzhev algebra.

By Theorem 5.1.8, this conjecture would yield the Jacobian Conjecture.

5.1.2.2. The case of binary algebras. When A is a binary algebra, *Engel type* means that the left and right multiplication mappings are both nilpotent.

A well-known result of S. Wang [21] shows that the Jacobian Conjecture holds for quadratic mappings

$$\mathbf{x} \rightarrow \mathbf{x} + \Psi_2(\mathbf{x}, \mathbf{x}).$$

If two different points (x_1, \dots, x_n) and (y_1, \dots, y_n) of an affine space are mapped to the same point by (f_1, \dots, f_n) , then the fact that the vertex of a parabola is in the middle of the interval whose endpoints are at the roots shows that all $f_i(\mathbf{x})$ have gradients at this midpoint $P = (\mathbf{x} + \mathbf{y})/2$ perpendicular to the line segment $[\mathbf{x}, \mathbf{y}]$. Hence the Jacobian is zero at the midpoint P . This fact holds in any characteristic $\neq 2$.

In Sec. 5.1.3 we prove the following theorem of Yagzhev, cf. Definition 5.1.15 below:

Theorem 5.1.10 (Yagzhev). *Every symmetric binary Engel type algebra of order k satisfying the system of Capelli identities of order n is weakly nilpotent, of weak nilpotence index bounded by some function $F(k, n)$.*

Remark 5.1.11. Yagzhev formulated his theorem as follows: *Every binary weakly Engel algebra of order k satisfying the system of Capelli identities of order n is weakly nilpotent, of index bounded by some function $F(k, n)$.* We obtain this reformulation, by replacing the algebra A by the algebra A^+ with multiplication given by $(a, b) = ab + ba$.

The following problems may help us understand the situation.

Problem. Obtain a straightforward proof of this theorem and deduce from it the Jacobian Conjecture for quadratic mappings.

Problem (generalized Jacobian Conjecture for quadratic mappings). Is every symmetric binary algebra of Engel type k , a Yagzhev algebra?

5.1.2.3. The case of ternary algebras. As we have observed, Yagzhev reduced the Jacobian Conjecture over a field of characteristic zero to the question:

Question 5.1.12. Is every finite dimensional ternary Engel algebra a Yagzhev algebra?

Drużkowski [84, 85] reduced this to the case when all cubic forms Ψ_{3i} are cubes of linear forms. Van den Essen and his school reduced the JC to the symmetric case (see [91, 92] for details). Bass, Connell, and Wright [21] use other methods including inversions. Yagzhev's approach matches that of [21], but using identities instead.

5.1.2.4. An example in nonzero characteristic of an Engel algebra that is not a Yagzhev algebra. Now we give an example, over an arbitrary field \mathbf{k} of characteristic $p > 3$, of a finite dimensional Engel algebra that is not a Yagzhev algebra, i.e., not weakly nilpotent. This means that the situation for binary algebras differs intrinsically from that for ternary algebras, and it would be worthwhile to understand why.

Theorem 5.1.13. *If $\text{char}(\mathbf{k}) = p > 3$, then there exists a finite dimensional \mathbf{k} -algebra that is Engel but not weakly nilpotent.*

Proof. Consider the noninvertible mapping $F : \mathbf{k}[x] \rightarrow \mathbf{k}[x]$ with Jacobian 1:

$$F : x \rightarrow x + x^p.$$

We introduce new commuting indeterminates $\{y_i\}_{i=1}^n$ and extend this mapping to $\mathbf{k}[x, y_1, \dots, y_n]$ by sending $y_i \mapsto y_i$. If n is big enough, then it is possible to find tame automorphisms G_1 and G_2 such that $G_1 \circ F \circ G_2$ is a cubic mapping $\mathbf{x} \rightarrow \mathbf{x} + \Psi_3(\mathbf{x})$, as follows:

Suppose we have a mapping

$$F : x_i \rightarrow P(x) + M$$

where $M = t_1 t_2 t_3 t_4$ is a monomial of degree at least 4. Introduce two new commuting indeterminates z, y and take $F(z) = z$, $F(y) = y$.

Define the mapping G_1 via $G_1(z) = z + t_1 t_2$, $G_1(y) = y + t_3 t_4$ with G_1 fixing all other indeterminates; define G_2 via $G_2(x) = x - yz$ with G_2 fixing all other indeterminates.

The composite mapping $G_1 \circ F \circ G_2$ sends x to $P(x) - yz - y t_1 t_2 - z t_3 t_4$, y to $y + t_3 t_4$, z to $z + t_1 t_2$, and agrees with F on all other indeterminates.

Note that we have removed the monomial $M = t_1 t_2 t_3 t_4$ from the image of F , but instead have obtained various monomials of smaller degree ($t_1 t_2$, $t_3 t_4$, zy , $z t_3 t_4$, $y t_1 t_2$). It is easy to see that this process terminates.

Our new mapping $H(x) = x + \Psi_2(x) + \Psi_3(x)$ is noninvertible and has Jacobian 1. Consider its blowup

$$R : x \mapsto x + T^2 y + T \Psi_2(x), \quad y \mapsto y - \Psi_3(x), \quad T \mapsto T.$$

This mapping R is invertible if and only if the initial mapping is invertible, and has Jacobian 1 if and only if the initial mapping has Jacobian 1, by [220, Lemma 2]. This mapping is also cubic homogeneous. The corresponding ternary algebra is Engel, but not weakly nilpotent. \square

This example shows that a direct combinatorial approach to the Jacobian Conjecture encounters difficulties, and in working with related Burnside type problems (in the sense of Zelmanov [225], dealing with nilpotence properties of Engel algebras, as indicated in the introduction), one should take into account specific properties arising in characteristic zero.

Definition 5.1.14. An algebra A is *nilpotent of class $\leq n$* if $M(a_1, a_2, \dots) = 0$ for each monomial $M(x_1, x_2, \dots)$ of degree $\geq n$. An ideal I of A is *strongly nilpotent of class $\leq n$* if $M(a_1, a_2, \dots) = 0$ for each monomial $M(x_1, x_2, \dots)$ in which indeterminates of total degree $\geq n$ have been substituted to elements of I .

Although the notions of nilpotent and strongly nilpotent coincide in the associative case, they differ for ideals of nonassociative algebras. For example, consider the following algebra suggested by Shestakov: A is the algebra generated by a, b, z satisfying the relations $a^2 = b$, $bz = a$ and all other

products 0. Then $I = Fa + Fb$ is nilpotent as a subalgebra, satisfying $I^3 = 0$ but not strongly nilpotent (as an ideal), since

$$b = ((a(bz))z)a \neq 0,$$

and one can continue indefinitely in this vein. Also, [107] contains an example of a finite dimensional non-associative algebra without any ideal which is maximal with respect to being nilpotent as a subalgebra.

In connection with the Generalized Jacobian Conjecture in characteristic 0, it follows from results of Yagzhev [222], also cf. [97], that there exists a 20-dimensional Engel algebra over \mathbb{Q} , not weakly nilpotent, satisfying the identities

$$x^2y = -yx^2, \quad (((yx^2)x^2)x^2)x^2 = 0, \quad (xy + yx)y = 2y^2x, \quad x^2y^2 = 0.$$

However, this algebra can be seen to be Yagzhev (see Definition 5.1.5).

For associative algebras, one uses the term “nil” instead of “weakly nilpotent.” Any nil subalgebra of a finite dimensional associative algebra is nilpotent, by Wedderburn’s Theorem [207]). Jacobson generalized this result to other settings, cf. [159, Theorem 15.23], and Shestakov [179] generalized it to a wide class of Jordan algebras (not necessarily commutative).

Yagzhev’s investigation of weak nilpotence has applications to the Koethe Conjecture, for algebras over uncountable fields. He reproved the following fact: *In every associative algebra over an uncountable field, the sum of every two nil right ideals is a nil right ideal.* This fact was proved first by Amitsur [3]. Amitsur’s result is for affine algebras, but one can easily reduce to the affine case.

5.1.2.5. Algebras satisfying systems of Capelli identities.

Definition 5.1.15. The *Capelli polynomial* C_k of order k is

$$C_k := \sum_{\sigma \in S_k} (-1)^\sigma x_{\sigma(1)} y_1 \cdots x_{\sigma(k)} y_k.$$

It is obvious that an associative algebra satisfies the Capelli identity c_k if and only if, for any monomial $M(x_1, \dots, x_k, y_1, \dots, y_r)$ multilinear in the x_i , the following equation holds identically in A :

$$\sum_{\sigma \in S_k} (-1)^\sigma M(v_{\sigma(1)}, \dots, v_{\sigma(k)}, y_1, \dots, y_r) = 0. \quad (5.1.8)$$

However, this does not apply to nonassociative algebras, so we need to generalize this condition.

Definition 5.1.16. The algebra A satisfies a *system of Capelli identities* of order k , if (5.1.8) holds identically in A for any monomial $M(x_1, \dots, x_k, y_1, \dots, y_r)$ multilinear in the x_i .

Any algebra of dimension $< k$ over a field satisfies a system of Capelli identities of order k . Algebras satisfying systems of Capelli identities behave much like finite dimensional algebras. They were introduced and systematically studied by Rasmuslov [155, 156].

Using Rasmuslov’s method, Zubrilin proved (see [231] and also [157, 229]) that if A is an arbitrary algebra satisfying the system of Capelli identities of order n , then the chain of ideals defining the *solvable radical* stabilizes at the n th step. More precisely, we utilize a Baer-type radical, along the lines of Amitsur [4].

Given an algebra A , we define

$$\text{Solv}_1 := \text{Solv}_1(A) = \sum \{\text{strongly nilpotent ideals of } A\},$$

and inductively, given Solv_k , define Solv_{k+1} by $\text{Solv}_{k+1} / \text{Solv}_k = \text{Solv}_1(A / \text{Solv}_k)$. For a limit ordinal α , define

$$\text{Solv}_\alpha = \bigcup_{\beta < \alpha} \text{Solv}_\beta.$$

This must stabilize at some ordinal α , for which we define $\text{Solv}(A) = \text{Solv}_\alpha$.

Clearly, $\text{Solv}(A/\text{Solv}(A)) = 0$; i.e., $A/\text{Solv}(A)$ has no nonzero strongly nilpotent ideals. Actually, Amitsur [4] defines $\zeta(A)$ as built up from ideals having trivial multiplication, and proves [4, Theorem 1.1] that $\zeta(A)$ is the intersection of the prime ideals of A .

We shall use the notion of *sandwich*, introduced by Kaplansky and Kostrikin, which is a powerful tool for Burnside type problems [225]. An ideal I is called a *sandwich ideal* if, for any k ,

$$M(z_1, z_2, x_1, \dots, x_k) = 0$$

for any $z_1, z_2 \in I$, any set of elements x_1, \dots, x_k , and any multilinear monomial M of degree $k + 2$. (Similarly, if the operations of an algebra have degree $\leq \ell$, then it is natural to use ℓ -sandwiches, which by definition satisfy the property that

$$M(z_1, \dots, z_\ell, x_1, \dots, x_k) = 0$$

for any $z_1, \dots, z_\ell \in I$, any set of elements x_1, \dots, x_k , and any multilinear monomial M of degree $k + \ell$.)

The next useful lemma follows from a result from [231]:

Lemma 5.1.17. *If an ideal I is strongly nilpotent of class ℓ , then there exists a decreasing sequence of ideals $I = I_1 \supseteq \dots \supseteq I_{l+1} = 0$ such that I_s/I_{s+1} is a sandwich ideal in A/I_{s+1} for all $s \leq l$.*

Definition 5.1.18. An algebra A is *representable* if it can be embedded into an algebra finite dimensional over some extension of the ground field.

Remark 5.1.19. Zubrilin proved (see [231]) a more precise statement, namely, if an algebra A of arbitrary signature satisfies a system of Capelli identities C_{n+1} , then there exists a sequence $B_0 \subseteq B_1 \subseteq \dots \subseteq B_n$ of strongly nilpotent ideals such that:

- (i) the natural projection of B_i in A/B_{i-1} is a strongly nilpotent ideal;
- (ii) A/B_n is representable;
- (iii) if $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n$ is any sequence of ideals of A such that I_{j+1}/I_j is a sandwich ideal in A/I_j , then $B_n \supseteq I_n$.

Such a sequence of ideals will be called a *Baer–Amitsur sequence*. In affine space the Zariski closure of the radical is radical, and hence the factor algebra is representable. (Although the radical coincides with the linear closure if the base field is infinite (see [37]), this assertion holds for arbitrary signatures and base fields.) Hence in representable algebras, the Baer–Amitsur sequence stabilizes after finitely many steps. Lemma 5.1.17 follows from these considerations.

Our next main goal is to prove Theorem 5.1.22 below, but first we need another notion.

5.1.2.6. The tree associated to a monomial. Effects of nilpotence have been used by different authors in another language. We associate a *rooted labelled tree* to any monomial: Any branching vertex indicates the symbol of an operator, whose outgoing edges are the terms in the corresponding symbol. Here is the precise definition.

Definition 5.1.20. Let $M(x_1, \dots, x_n)$ be a monomial in an algebra A of arbitrary signature. One can associate the tree T_M by an inductive procedure:

- (i) If M is a single variable, then T_M is just the vertex \bullet .
- (ii) Let $M = g(M_1, \dots, M_k)$, where g is a k -ary operator. We assume inductively that the trees T_i , $i = 1, \dots, k$, are already defined. Then the tree T_M is the disjoint union of the T_i , together with the root \bullet and arrows starting with \bullet and ending with the roots of the trees T_i .

Remark 5.1.21. Sometimes one labels T_M according to the operator g and the positions inside g .

If the outgoing degree of each vertex is 0 or 2, the tree is called *binary*. If the outgoing degree of each vertex is either 0 or 3, the tree is called *ternary*. If each operator is binary, T_M will be binary; if each operator is ternary, T_M will be ternary.

5.1.3. Lifting Yagzhev algebras. Recall Definitions 5.1.5 and 5.1.7.

Theorem 5.1.22. *Suppose A is an algebra of Engel type, and let I be a sandwich ideal of A . If A/I is Yagzhev, then A is Yagzhev.*

The proof follows easily from the following two propositions.

Let k be the class of weak nilpotence of A/I . We call a branch of the tree *fat* if it has more than k entries.

Proposition 5.1.23.

- (a) *The sum of all monomials of any degree $s > k$ belongs to I .*
- (b) *Let x_1, \dots, x_n be fixed indeterminates and M be an arbitrary monomial with $s_1, \dots, s_\ell > k$. Then*

$$\sum_{\substack{|t_1|=s_1, \dots, \\ |t_\ell|=s_\ell}} M(x_1, \dots, x_n, t_1, \dots, t_\ell) \equiv 0. \quad (5.1.9)$$

- (c) *The sum of all monomials of degree s , containing at least ℓ nonintersecting fat branches, is zero.*

Proof. Assertion (a) is just a reformulation of the weak nilpotence of A/I ; (b) follows from (a) and the sandwich property of an ideal I . To get (c) from (b), it suffices to consider the highest nonintersecting fat branches. \square

Proposition 5.1.24 (Yagzhev). *The linearization of the sum of all terms with a fixed fat branch of length n is the complete linearization of the function*

$$\sum_{\sigma \in S_n} \prod (\text{Ad}_{k_{\sigma(i)}})(z)(t).$$

Theorem 1.2, Lemma 5.1.17, and Zubrilin's result give us the following major result.

Theorem 5.1.25. *In characteristic zero, the Jacobian conjecture is equivalent to the following statement: Any algebra of Engel type satisfying some system of Capelli identities is a Yagzhev algebra.*

This theorem generalizes the following result of Yagzhev.

Theorem 5.1.26. *The Jacobian conjecture is equivalent to the following statement: Any ternary Engel algebra in characteristic 0 satisfying a system of Capelli identities is a Yagzhev algebra.*

The Yagzhev correspondence and the results of this section (in particular, Theorem 5.1.25) yield the proof of Theorem 5.1.10.

5.1.3.1. Sparse identities. Generalizing Capelli identities, we say that an algebra satisfies a system of *sparse identities* when there exist k and coefficients α_σ such that for any monomial $M(x_1, \dots, x_k, y_1, \dots, y_r)$ multilinear in x_i , the following equation holds:

$$\sum_{\sigma} \alpha_{\sigma} M(c_1 v_{\sigma(1)} d_1, \dots, c_k v_{\sigma(k)} d_k, y_1, \dots, y_r) = 0. \quad (5.1.10)$$

Note that one need only check (5.1.10) for monomials. The system of Capelli identities is a special case of a system of sparse identities (when $\alpha_{\sigma} = (-1)^{\sigma}$). This concept ties in with the following “few long branches” lemma [230], concerning the structure of trees of monomials for algebras with sparse identities:

Lemma 5.1.27 (Few long branches). *Suppose an algebra A satisfies a system of sparse identities of order m . Then any monomial is linearly representable by monomials such that the corresponding tree has not more than $m - 1$ disjoint branches of length $\geq m$.*

Lemma 5.1.27 may be useful in studying nilpotence of Engel algebras.

5.1.4. Inversion formulas and problems of Burnside type. We have seen that the JC relates to problems of “Specht type” (concerning whether one set of polynomial identities implies another), as well as problems of Burnside type.

Burnside type problems become more complicated in nonzero characteristic; cf. Zelmanov’s review paper [225].

Bass, Connell, and Wright [21] attacked the JC by means of inversion formulas. D. Wright [208] wrote an inversion formula for the symmetric case and related it to a combinatorial structure called the *Grossman–Larson Algebra*. Namely, write $F = X - H$, and define $J(H)$ to be the Jacobian matrix of H . Wright proved the JC for the case where H is homogeneous and $J(H)^3 = 0$, and also for the case where H is cubic and $J(H)^4 = 0$; these correspond in Yagzhev’s terminology to the cases of Engel type 3 and 4, respectively. Also, the so-called *chain vanishing theorem* in [208] follows from Engel type. Similar results were obtained earlier by Singer [186] using tree formulas for formal inverses. The inversion formula, introduced in [21], was investigated by D. Wright and his school. Many authors use the language of so-called *tree expansion* (see [186, 208] for details). In view of Theorem 5.1.13, the tree expansion technique should be highly nontrivial.

The Jacobian Conjecture can be formulated as a question of quantum field theory (see [1]), in which tree expansions are seen to correspond to Feynmann diagrams.

In [186, 208] (see also [209]), trees with one label correspond to elements of the algebra A built by Yagzhev, and 2-labelled trees correspond to the elements of the operator algebra $D(A)$ (the algebra generated by operators $x \rightarrow M(x, \mathbf{y})$, where M is some monomial). These authors deduce weak nilpotence from the Engel conditions of degree 3 and 4. The inversion formula for automorphisms of tensor product of Weyl algebras and the ring of polynomials was studied intensively in the papers [27, 30]. Using techniques from [42], this yields a slightly different proof of the equivalence between the JC and DC, by an argument similar to one given in [223]. Yagzhev’s approach makes the situation much clearer, and the known approaches to the Jacobian Conjecture using inversion formulas can be explained from this viewpoint.

Remark 5.1.28. The most recent inversion formula (and probably the most algebraically explicit one) was obtained by V. Bavula [26]. The coefficient q_0 can be made explicit in (5.1.5), by means of the Gabber Inequality, which says that if

$$f : K^n \rightarrow K_n; \quad x_i \rightarrow f_i(\mathbf{x})$$

is a polynomial automorphism, with $\deg(f) = \max_i \deg(f_i)$, then $\deg(f^{-1}) \leq \deg(f)^{n-1}$

In fact, we are working with *operads*, cf. the classical book [146]. A review of operad theory and its relation with physics and PI-theory in particular Burnside type problems, will appear in D. Piontkovsky [153] (see also [109, 154]). Operad theory provides a supply of natural identities and varieties, but they also correspond to geometric facts. For example, the Jacobi identity corresponds to the fact that the altitudes of a triangle are concurrent. M. Dehn’s observations that the Desargue property of a projective plane corresponds to associativity of its coordinate ring, and Pappus’ property to its commutativity, can be considered as a first step in operad theory. Operads are important in mathematical physics, and formulas for the famous Kontsevich quantization theorem resemble formulas for the inverse mapping. The operators considered here are operads.

5.2. THE JACOBIAN CONJECTURE FOR VARIETIES, AND DEFORMATIONS

In this section we consider analogs of the JC for other varieties of algebras, partially with the aim on throwing light on the classical JC (for the commutative associative polynomial algebra).

5.2.1. Generalization of the Jacobian Conjecture to arbitrary varieties. J. Birman [48] already proved the JC for free groups in 1973. The JC for free associative algebras (in two generators) was established in 1982 by W. Dicks and J. Levin [73, 74], utilizing Fox derivatives, which we describe later on. Their result was reproved by Yagzhev [218], whose ideas are sketched in this section. Also see Schofield [175], who proved the full version. Yagzhev then applied these ideas to other varieties of algebras [217, 222] including nonassociative commutative algebras and anti-commutative algebras;

U. U. Umirbaev [195] generalized these to “Schreier varieties,” defined by the property that every subalgebra of a free algebra is free. The JC for free Lie algebras was proved by Reutenauer [158], Shpilrain [185], and Umirbaev [194].

The Jacobian Conjecture for varieties generated by finite dimensional algebras, is closely related to the Jacobian Conjecture in the usual commutative associative case, which is the most important.

Let \mathfrak{M} be a variety of algebras of some signature Ω over a given field \mathbf{k} of characteristic zero, and $\mathbf{k}_{\mathfrak{M}}\langle \mathbf{x} \rangle$ the relatively free algebra in \mathfrak{M} with generators $\mathbf{x} = \{x_i : i \in I\}$. We assume that $|\Omega|, |I| < \infty$, $I = 1, \dots, n$.

Take a set $\mathbf{y} = \{y_i\}_{i=1}^n$ of new indeterminates. For any $f(\mathbf{x}) \in \mathbf{k}_{\mathfrak{M}}\langle \mathbf{x} \rangle$ one can define an element $\hat{f}(\mathbf{x}, \mathbf{y}) \in \mathbf{k}_{\mathfrak{M}}\langle \mathbf{x}, \mathbf{y} \rangle$ via the equation

$$f(x_1 + y_1, \dots, x_n + y_n) = f(\mathbf{y}) + \hat{f}(\mathbf{x}, \mathbf{y}) + R(\mathbf{x}, \mathbf{y}) \quad (5.2.1)$$

where $\hat{f}(\mathbf{x}, \mathbf{y})$ has degree 1 with respect to \mathbf{x} , and $R(\mathbf{x}, \mathbf{y})$ is the sum of monomials of degree ≥ 2 with respect to \mathbf{x} ; \hat{f} is a generalization of the differential.

Let $\alpha \in \text{End}(\mathbf{k}_{\mathfrak{M}}\langle \mathbf{x} \rangle)$, i.e.,

$$\alpha : x_i \mapsto f_i(\mathbf{x}); \quad i = 1, \dots, n. \quad (5.2.2)$$

Definition 5.2.1. Define the Jacobi endomorphism $\hat{\alpha} \in \text{End}(\mathbf{k}_{\mathfrak{M}}\langle \mathbf{x}, \mathbf{y} \rangle)$ via the equality

$$\hat{\alpha} : \begin{cases} x_i \rightarrow \hat{f}_i(\mathbf{x}), \\ y_i \rightarrow y_i. \end{cases} \quad (5.2.3)$$

The Jacobi mapping $f \mapsto \hat{f}$ satisfies the chain rule, in the sense that it preserves composition.

Remark 5.2.2. It is not difficult to check (and is well known) that if $\alpha \in \text{Aut}(\mathbf{k}_{\mathfrak{M}}\langle \mathbf{x} \rangle)$ then $\hat{\alpha} \in \text{Aut}(\mathbf{k}_{\mathfrak{M}}\langle \mathbf{x}, \mathbf{y} \rangle)$.

The inverse implication is called the *Jacobian Conjecture for the variety \mathfrak{M}* . Here is an important special case.

Definition 5.2.3. Let $A \in \mathfrak{M}$ be a finite dimensional algebra, with base $\{e_i\}_{i=1}^N$. Consider a set of commutative indeterminates $\boldsymbol{\nu} = \{\nu_{si} | s = 1, \dots, n; i = 1, \dots, N\}$. The elements

$$z_j = \sum_{i=1}^N \nu_{ji} e_i; \quad j = 1, \dots, n,$$

are called *generic elements of A* .

Usually in the matrix algebra $\mathbb{M}_m(\mathbf{k})$, the set of matrix units $\{e_{ij}\}_{i,j=1}^m$ is taken as the base. In this case $e_{ij}e_{kl} = \delta_{jk}e_{il}$ and

$$z_l = \sum_{ij} \lambda_{ij}^l e_{ij}, \quad l = 1, \dots, n.$$

Definition 5.2.4. A *generic matrix* is a matrix whose entries are distinct commutative indeterminates, and the so-called *algebra of generic matrices of order m* is generated by associative generic $m \times m$ matrices.

The algebra of generic matrices is prime, and every prime, relatively free, finitely generated associative *PI*-algebra is isomorphic to an algebra of generic matrices. If we include taking traces as an operator in the signature, then we get the *algebra of generic matrices with trace*. That algebra is a Noetherian module over its center.

Define the \mathbf{k} -linear mappings

$$\Omega_i : \mathbf{k}_{\mathfrak{M}}\langle \mathbf{x} \rangle \rightarrow \mathbf{k}[\boldsymbol{\nu}]; \quad i = 1, \dots, n$$

via the relation

$$f\left(\sum_{i=1}^N \nu_{1i} e_i, \dots, \sum_{i=1}^N \nu_{ni} e_i\right) = \sum_{i=1}^N (f\Omega_i) e_i.$$

It is easy to see that the polynomials $f\Omega_i$ are uniquely determined by f .

One can define the mapping

$$\varphi_A : \text{End}(k_{\mathfrak{M}} \langle \mathbf{x} \rangle) \rightarrow \text{End}(k[\boldsymbol{\nu}])$$

as follows. If

$$\alpha \in \text{End}(k_{\mathfrak{M}} \langle \mathbf{x} \rangle) : x_s \rightarrow f_s(\mathbf{x}) \quad s = 1, \dots, n,$$

then $\varphi_A(\alpha) \in \text{End}(k[\boldsymbol{\nu}])$ can be defined via the relation

$$\varphi_A(\alpha) : \nu_{si} \rightarrow P_{si}(\boldsymbol{\nu}); \quad s = 1, \dots, n; \quad i = 1, \dots, n,$$

where $P_{si}(\boldsymbol{\nu}) = f_s \Omega_i$.

The following proposition is well known.

Proposition 5.2.5 (see [222]). *Let $A \in \mathfrak{M}$ be a finite dimensional algebra, and $\mathbf{x} = \{x_1, \dots, x_n\}$ be a finite set of commutative indeterminates. Then the mapping φ_A is a semigroup homomorphism, sending 1 to 1, and automorphisms to automorphisms. Also the mapping φ_A commutes with the operation $\widehat{}$ of taking the Jacobi endomorphism, in the sense that $\widehat{\varphi_A(\alpha)} = \varphi_A(\widehat{\alpha})$. If φ is invertible, then $\widehat{\varphi}$ is also invertible.*

This proposition is important, since as noted after Remark 5.2.2, the opposite direction is the JC.

5.2.2. Deformations and the Jacobian Conjecture for free associative algebras.

Definition 5.2.6. A T -ideal is a completely characteristic ideal, i.e., stable under any endomorphism.

Proposition 5.2.7. *Suppose A is a relatively free algebra in the variety \mathfrak{M} , I is a T -ideal in A , and $\mathfrak{M}' = \text{Var}(A/I)$. Any polynomial mapping $F : A \rightarrow A$ induces a natural mapping $F' : A/I \rightarrow A/I$, as well as a mapping $\widehat{F'}$ in \mathfrak{M}' . If F is invertible, then F' is invertible; if \widehat{F} is invertible, then $\widehat{F'}$ is also invertible.*

For example, let F be a polynomial endomorphism of the free associative algebra $k \langle \mathbf{x} \rangle$, and I_n be the T -ideal of the algebra of generic matrices of order n . Then $F(I_n) \subseteq I_n$ for all n . Hence F induces an endomorphism F_{I_n} of $k \langle \mathbf{x} \rangle / I_n$. In particular, this is a semigroup homomorphism. Thus, if F is invertible, then F_{I_n} is invertible, but not vice versa.

The Jacobian mapping $\widehat{F_{I_n}}$ of the reduced endomorphism F_{I_n} is the reduction of the Jacobian mapping of F .

5.2.2.1. The Jacobian Conjecture and the packing property. This subsection is based on the *packing property* and deformations. Let us illustrate the main idea. It is well known that the composite of ALL quadratic extensions of \mathbb{Q} is infinite dimensional over \mathbb{Q} . Hence all such extensions cannot be embedded (“packed”) into a single commutative finite dimensional \mathbb{Q} -algebra. However, all of them can be packed into $\mathbb{M}_2(\mathbb{Q})$. We formalize the notion of packing in §5.2.5. Moreover, for ANY elements NOT in \mathbb{Q} there is a parametric family of embeddings (because it embeds non-centrally and thus can be deformed via conjugation by a parametric set of matrices). Uniqueness thus means belonging to the center. Similarly, adjoining noncommutative coefficients allows one to decompose polynomials, as to be elaborated below.

This idea allows us to solve equations via a finite dimensional extension, and to find a parametric sets of solutions if some solution does not belong to the original algebra. That situation contradicts local invertibility.

Let F be an endomorphism of the free associative algebra having invertible Jacobian. We suppose that $F(0) = 0$ and

$$F(x_i) = x_i + \sum \text{terms of order } \geq 2.$$

We intend to show how the invertibility of the Jacobian implies invertibility of the mapping F .

Let Y_1, \dots, Y_k be generic $m \times m$ matrices. Consider the system of equations

$$\{F_i(X_1, \dots, X_n) = Y_i; \quad i = 1, \dots, k\}.$$

This system has a solution over some finite extension of order m of the field generated by the center of the algebra of generic matrices *with trace*.

Consider the set of block diagonal $mn \times mn$ matrices:

$$A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & & A_n \end{pmatrix}, \quad (5.2.4)$$

where the A_j are $m \times m$ matrices.

Next, we consider the system of equations

$$\{F_i(X_1, \dots, X_n) = Y_i; i = 1, \dots, k\}, \quad (5.2.5)$$

where the $mn \times mn$ matrices Y_i have the form (5.2.4) with the A_j generic matrices.

Any m -dimensional extension of the base field \mathbf{k} is embedded into $\mathbb{M}_m(\mathbf{k})$. But $\mathbb{M}_{mn}(\mathbf{k}) \simeq \mathbb{M}_m(\mathbf{k}) \otimes \mathbb{M}_n(\mathbf{k})$. It follows that for appropriate m , the system (5.2.5) has a unique solution in the matrix ring with traces. (Each is given by a matrix power series where the summands are matrices whose entries are homogeneous forms, seen by rewriting $Y_i = X_i + \sum$ terms of order 2 as $X_i = Y_i + \sum$ terms of order 2, and iterating.) The solution is unique since their entries are distinct commuting indeterminates.

If F is invertible, then this solution must have block diagonal form. However, if F is not invertible, this solution need not have block diagonal form. Now we translate invertibility of the Jacobian to the language of *parametric families* or *deformations*.

Consider the matrices

$$E_\lambda^\ell = \begin{pmatrix} E & 0 & \dots & 0 \\ 0 & \ddots & & \dots & 0 \\ 0 & \dots & \lambda \cdot E & & 0 \\ \vdots & & \dots & \ddots & \vdots \\ 0 & & \dots & & E \end{pmatrix}$$

where E denotes the identity matrix. (The index ℓ designates the position of the block $\lambda \cdot E$.) Taking X_j not to be a block diagonal matrix, then for some ℓ we obtain a non-constant parametric family $E_\lambda^\ell X_j (E_\lambda^\ell)^{-1}$ dependent on λ .

On the other hand, if Y_i has form (5.2.4), then $E_\lambda^\ell Y_i (E_\lambda^\ell)^{-1} = Y_i$ for all $\lambda \neq 0$; $\ell = 1, \dots, k$.

Hence, if F_{I_n} is not an automorphism, then we have a *continuous parametric set of solutions*. But if the Jacobian mapping is invertible, it is locally 1:1, a contradiction. This argument yields the following result:

Theorem 5.2.8. *For $F \in \text{End}(k\langle \mathbf{x} \rangle)$, if the Jacobian of F is invertible, then the reduction F_{I_n} of F , modulo the T -ideal of the algebra of generic matrices, is invertible.*

For details of the proof, see [217]. Because any relatively free affine algebra of characteristic 0 satisfies the set of identities of some matrix algebra, it is the quotient of the algebra of generic matrices by some T -ideal J . But J maps into itself after any endomorphism of the algebra. We conclude the following fact.

Corollary 5.2.9. *If $F \in \text{End}(k\langle \mathbf{x} \rangle)$ and the Jacobian of F is invertible, then the reduction F_J of F modulo any proper T -ideal J is invertible.*

In order to get invertibility of \mathbf{F} itself, Yagzhev used the additional ideas:

1. The block diagonal technique works equally well on skew fields.
2. The above algebraic constructions can be carried out on Ore extensions, in particular for the *Weyl algebras* $W_n = \mathbf{k}[x_1, \dots, x_n; \partial_1, \dots, \partial_n]$.
3. By a result of L. Makar-Limanov, the free associative algebra can be embedded into the ring of fractions of the Weyl algebra. This provides a nice presentation for mapping the free algebra.

Definition 5.2.10. Let A be an algebra, $B \subset A$ a subalgebra, and $\alpha : A \rightarrow A$ a polynomial mapping of A (and hence $\alpha(B) \subset B$, see Definition 5.1.1). B is a *test algebra* for α , if $\alpha(A \setminus B) \neq A \setminus B$.

The next theorem shows the universality of the notion of a test algebra. An endomorphism is called *rationally invertible* if it is invertible over Cohn's skew field of fractions [65] of $\mathbf{k}\langle \mathbf{x} \rangle$.

Theorem 5.2.11 (Yagzhev). *For any $\alpha \in \text{End}(\mathbf{k}\langle \mathbf{x} \rangle)$, one of the two statements holds:*

- (i) α is rationally invertible, and its reduction to any finite-dimensional factor also is rationally invertible;
- (ii) there exists a test algebra for some finite dimensional reduction of α .

This theorem implies the Jacobian conjecture for free associative algebras. We do not go into details, referring the reader to the papers [217, 222].

Remark 5.2.12. The same idea is used in quantum physics. The polynomial $x^2 + y^2 + z^2$ cannot be decomposed for any commutative ring of coefficients. However, it can be decomposed using noncommutative ring of coefficients (Pauli matrices). The Laplace operator in 3-dimensional space can be decomposed in such a manner.

5.2.2.2. Reduction to nonzero characteristic. One can work with deformations equally well in nonzero characteristic. However, the naive Jacobian condition does not give us parametric families, because of consequences of inseparability. Hence it is interesting using deformations to get a reasonable version of the JC for characteristic $p > 0$, especially because of recent progress in the JC related to the reduction of holonomic modules to the case of characteristic p and investigation of the p -curvature or Poisson brackets on the center (see [41, 42, 190]).

In his very last paper [223], A. V. Yagzhev approached the JC using positive characteristics. He noted that the existence of a counterexample is equivalent to the existence of an Engel, but not Yagzhev, finite dimensional ternary algebra in each positive characteristic $p \gg 0$. (This fact was also used in [41, 42, 190].)

If a counterexample to the JC exists, then such an algebra A exists even over a finite field, and hence can be finite. It generates a locally finite variety of algebras that are of Engel type, but not Yagzhev. This situation can be reduced to the case of a locally semiprime variety. Any relatively free algebra of this variety is semiprime, and the centroid of its localization is a finite direct sum of fields. The situation can be reduced to one field, and he tried to construct an embedding which is not an automorphism. This would contradict the finiteness property.

Since a reduction of an endomorphism as a mapping on points of finite height may be an automorphism, the issue of injectivity also arises. However, this approach looks promising, and may involve new ideas, such as in [41, 42, 190]. Perhaps different infinitesimal conditions (like the Jacobian condition in characteristic zero) can be found.

5.2.3. The Jacobian Conjecture for other classes of algebras. Although the Jacobian Conjecture remains open for commutative associative algebras, it has been established for other classes of algebras, including free associative algebras, free Lie algebras, and free metabelian algebras (for further details, see Sec. 5.2.1).

An algebra is said to be *metabelian* if it satisfies the identity $[x, y][z, t] = 0$.

The case of free metabelian algebras, established by Umirbaev (see [193]), involves some interesting new ideas that we describe now. His method of proof is by means of co-multiplication, taken from the theory of Hopf algebras and quantization. Let A^{op} denote the opposite algebra of the free associative algebra A , with generators t_i . For $f \in A$ we denote the corresponding element of A^{op} as f^* . Consider the mapping

$$\lambda : A^{op} \otimes A \rightarrow A, \quad \lambda \left(\sum f_i^* \otimes g_i \right) = \sum f_i g_i.$$

Then $I_A := \ker(\lambda)$ is a free A bimodule with generators $t_i^* \otimes 1 - 1 \otimes t_i$. The mapping

$$d_A : A \rightarrow I_A, \quad d_A(a) = a^* \otimes 1 - 1 \otimes a$$

is called the *universal derivation* of A . The *Fox derivatives* $\partial a / \partial t_i \in A^{op} \otimes A$ (see [94]) are defined as

$$d_A(a) = \sum_i (t_i^* \otimes 1 - 1 \otimes t_i) \frac{\partial a}{\partial t_i}$$

(cf. [74, 193]).

Let $C = A / \text{Id}([A, A])$ be a free commutative associative algebra and $B = A / \text{Id}([A, A])^2$ be a free metabelian algebra. Let

$$\partial(a) = \left(\frac{\partial a}{\partial t_1}, \dots, \frac{\partial a}{\partial t_n} \right).$$

One can define the natural derivations

$$\bar{\partial} : A \rightarrow (A' \otimes A)^n \rightarrow (C' \otimes C)^n, \quad \tilde{\partial} : A \rightarrow (C' \otimes C)^n \rightarrow C^n, \quad (5.2.6)$$

where the mapping $(C' \otimes C)^n \rightarrow C^n$ is induced by λ . Then

$$\ker(\bar{\partial}) = \text{Id}([A, A])^2 + F$$

and $\bar{\partial}$ induces a derivation $B \rightarrow (C' \otimes C)^n$, whereas $\tilde{\partial}$ induces the usual derivation $C \rightarrow C^n$. Let $\Delta : C \rightarrow C' \otimes C$ be the mapping induced by d_A , i.e.,

$$\Delta(f) = f^* \otimes 1 - 1 \otimes f,$$

and let $z_i = \Delta(x_i)$. The *Jacobi matrix* is defined in the natural way, and provides the formulation of the JC for free metabelian algebras. One of the crucial steps in proving the JC for free metabelian algebras is the following homological lemma from [193].

Lemma 5.2.13. *Let $\mathbf{u} = (u_1, \dots, u_n) \in (C^{op} \otimes C)^n$. Then $\mathbf{u} = \bar{\partial}(\bar{w})$ for some $w \in \text{Id}([A, A])$ if and only if*

$$\sum z_i u_i = 0.$$

The proof also requires the following theorem.

Theorem 5.2.14. *Let $\varphi \in \text{End}(C)$. Then $\varphi \in \text{Aut}(C)$ if and only if $\text{Id}(\Delta(\varphi(x_i)))_{i=1}^n = \text{Id}(z_i)_{i=1}^n$.*

The paper [193] also includes the following result.

Theorem 5.2.15. *Any automorphism of C can be extended to an automorphism of B , using the JC for the free metabelian algebra B .*

This is a nontrivial result, unlike the extension of an automorphism of B to an automorphism of $A / \text{Id}([A, A])^n$ for any $n > 1$.

5.2.4. Questions related to the Jacobian Conjecture. Let us turn to other interesting questions which can be linked to the Jacobian Conjecture. The quantization procedure is a bridge between the commutative and noncommutative cases and is deeply connected to the JC and related questions. Some of these questions also are discussed in the paper [81].

Relations between the free associative algebra and the classical commutative situation are very deep. In particular, Bergman's theorem that any commutative subalgebra of the free associative algebra is isomorphic to a polynomial ring in one indeterminate is the noncommutative analog of Zak's theorem [224] that any integrally closed subring of a polynomial ring of Krull dimension 1 is isomorphic to a polynomial ring in one indeterminate. For example, Bergman's theorem is used to describe the automorphism group $\text{Aut}(\text{End}(\mathbf{k}\langle x_1, \dots, x_n \rangle))$ (see [39]); Zak's theorem is used in the same way to describe the group $\text{Aut}(\text{End}(\mathbf{k}[x_1, \dots, x_n]))$ (see [43]).

Question 5.2.16. Can one prove Bergman's theorem via quantization?

Quantization could be a key idea for understanding Jacobian type problems in other varieties of algebras.

Cancellation problems. We recall the following three classical problems.

1. Let K_1 and K_2 be affine domains for which $K_1[t] \simeq K_2[t]$. Is it true that $K_1 \simeq K_2$?
2. Let K_1 and K_2 be an affine fields for which $K_1(t) \simeq K_2(t)$. Is it true that $K_1 \simeq K_2$? In particular, if $K(t)$ is a field of rational functions over the field \mathbf{k} , is it true that K is also a field of rational functions over \mathbf{k} ?
3. If $K[t] \simeq \mathbf{k}[x_1, \dots, x_n]$, is it true that $K \simeq \mathbf{k}[x_1, \dots, x_{n-1}]$?

In general, the answer to Problems 1 and 2 is negative (even if $\mathbf{k} = \mathbb{C}$; see [31, 36] and the references therein). However, Problem 2 has a positive solution in low dimensions. Problem 3 is currently called the *Cancellation Conjecture*, although Zariski's original cancellation conjecture was for fields (Problem 2). (For Zariski's conjecture and related problems, see [68, 105, 147, 187].) For $n \geq 3$, the Cancellation Conjecture (Problem 3) remains open, to the best of our knowledge, and it is reasonable to pose the Cancellation Conjecture for free associative rings and pose the following question.

Question 5.2.17. If $K * \mathbf{k}[t] \simeq \mathbf{k}\langle x_1, \dots, x_n \rangle$, then is $K \simeq \mathbf{k}\langle x_1, \dots, x_{n-1} \rangle$?

This question was solved for $n = 2$ by V. Drensky and J. T. Yu (see [79]).

2. The Tame Automorphism Problem. Yagzhev applied his approach to study the tame automorphism problem. Unfortunately, these papers have not survived.

It is easy to see that every endomorphism ϕ of a commutative algebra can be lifted to some endomorphism of the free associative algebra, and hence to some endomorphism of the algebra of generic matrices. However, it is not clear that any automorphism ϕ can be lifted to an automorphism.

We recall that an automorphism of $\mathbf{k}[x_1, \dots, x_n]$ is *elementary* if it has the form

$$x_1 \mapsto x_1 + f(x_2, \dots, x_n), \quad x_i \mapsto x_i, \quad \forall i \geq 2.$$

A *tame automorphism* is a product of elementary automorphisms, and a non-tame automorphism is called *wild*. The “tame automorphism problem” asks whether any automorphism is tame. Jung [103] and van der Kulk [203] proved this for $n = 2$, (also see [149, 150] for free groups, [65] for free Lie algebras, and [67, 141] for free associative algebras), so one takes $n > 2$.

Elementary automorphisms can be lifted to automorphisms of the free associative algebra; hence every tame automorphism can be so lifted. If an automorphism φ cannot be lifted to an automorphism of the algebra of generic matrices, it cannot be tame. This give us approach to the tame automorphism problem.

We can lift an automorphism of $\mathbf{k}[x_1, \dots, x_n]$ to an endomorphism of $\mathbf{k}\langle x_1, \dots, x_n \rangle$ in many ways. Then replacing x_1, \dots, x_n by generic $(N \times N)$ -matrices induces a polynomial mapping

$$F_{(N)} : \mathbf{k}^{nN^2} \rightarrow \mathbf{k}^{nN^2}.$$

For each automorphism φ , the invertibility of this mapping can be transformed into compatibility of some system of equations. For example, [152, Theorem 10.5] asserts that the Nagata automorphism is wild, provided that a certain system of five equations in 27 unknowns has no solutions. Whether Peretz' method can effectively attack tameness questions remains to be seen. The wildness of the Nagata automorphism was established by Shestakov and Umirbaev [183]. One important ingredient in the proof is *degree estimates* of an expression $p(f, g)$ of algebraically independent polynomials f and g in terms of the degrees of f and g , provided neither leading term is proportional to a power of the other, initiated by Shestakov and Umirbaev [182]. An exposition based on their method is given in Kuroda [106].

One of the most important tools is the degree estimation technique, which in the multidimensional case becomes the analysis of leading terms, and is more complicated. We refer to the deep papers [49, 106, 108]. Several papers of Kishimoto contain gaps, but also provide deep insights.

One can also ask the weaker question of “coordinate tameness:” Is the image of (x, y, z) under the Nagata automorphism the image under some (other) tame automorphism? This also fails, by [200].

An automorphism φ is called *stably tame* if, when several new indeterminates $\{t_i\}$ are adjoined, the extension of φ given by $\varphi'(t_i) = t_i$ is tame; otherwise it is called *stably wild*. Stable tameness of automorphisms of $\mathbf{k}[x, y, z]$ fixing z is proved in [47]; similar results for $\mathbf{k}\langle x, y, z \rangle$ are given in [44].

Yagzhev tried to construct wild automorphisms via polynomial automorphisms of the Cayley-Dickson algebra with base $\{e_i\}_{i=1}^8$, and the set $\{\nu_i, \xi_i, \varsigma_i\}_{i=1}^8$ of commuting indeterminates. Let

$$x = \sum \nu_i e_i, \quad y = \sum \xi_i e_i, \quad z = \sum \varsigma_i e_i.$$

Let (x, y, z) denote the associator $(xy)z - x(yz)$ of the elements x, y, z , and write

$$(x, y, z)^2 = \sum f_i(\nu, \xi, \varsigma) e_i.$$

Then the endomorphism G of the polynomial algebra given by

$$G : \nu_i \rightarrow \nu_i + f_i(\nu, \xi, \varsigma), \quad \xi_i \rightarrow \xi_i, \quad \varsigma_i \rightarrow \varsigma_i,$$

is an automorphism, which likely is stably wild.

In the free associative case, perhaps it is possible to construct an example of an automorphism, the wildness of which could be proved by considering its Jacobi endomorphism (Definition 5.2.1). He tried to find an automorphism α of an Yagzhev tried to construct examples of algebras $R = A \otimes A^{op}$ over which there are invertible matrices that cannot decompose as products of elementary ones. Yagzhev conjectured that the automorphism

$$x_1 \rightarrow x_1 + y_1(x_1y_2 - y_1x_2), \quad x_2 \rightarrow x_2 + (x_1y_2 - y_1x_2)y_2, \quad y_1 \rightarrow y_1, \quad y_2 \rightarrow y_2$$

of the free associative algebra is wild.

Umirbaev [196] proved in characteristic 0 that the *Anick automorphism* $x \rightarrow x + y(xy - yz)$, $y \rightarrow y$, $z \rightarrow z + (zy - yz)y$ is wild, by using metabelian algebras. The proof uses description of the defining relations of 3-variable automorphism groups [197–199]. Drensky and Yu [80, 82] proved in characteristic 0 that the image of x under the Anick Automorphism is not the image of any tame automorphism.

Stable Tameness Conjecture. *Every automorphism of the polynomial algebra $\mathbf{k}[x_1, \dots, x_n]$ (respectively, of the free associative algebra $\mathbf{k}\langle x_1, \dots, x_n \rangle$) is stably tame.*

Lifting in the free associative case is related to quantization. It provides some light on the similarities and differences between the commutative and noncommutative cases. Every tame automorphism of the polynomial ring can be lifted to an automorphism of the free associative algebra. There was a conjecture that *any wild z -automorphism of $\mathbf{k}[x, y, z]$ (i.e., fixing z) over an arbitrary field \mathbf{k} cannot be lifted to a z -automorphism of $\mathbf{k}\langle x, y, z \rangle$* . In particular, the Nagata automorphism cannot be so lifted [81]. This conjecture was solved by Belov and J.-T. Yu [38] over an arbitrary field. However, the general lifting conjecture is still open. In particular, it is not known whether the Nagata automorphism can be lifted to an automorphism of the free algebra. (Such a lifting could not fix z .)

The paper [38] describes all the z -automorphisms of $\mathbf{k}\langle x, y, z \rangle$ over an arbitrary field \mathbf{k} . Based on that work, Belov and J.-T. Yu [44] proved that every z -automorphism of $\mathbf{k}\langle x, y, z \rangle$ is stably tame, for all fields \mathbf{k} . A similar result in the commutative case is proved by Berson, van den Essen, and Wright [47]. These are important first steps towards solving the stable tameness conjecture in the noncommutative and commutative cases.

The free associative situation is much more rigid than the polynomial case. Degree estimates for the free associative case are the same for prime characteristic (see [137]) as in characteristic 0 (see [143]). The methodology is different from the commutative case, for which degree estimates (as well as examples of wild automorphisms) are not known in prime characteristic.

J.-T. Yu found some evidence of a connection between the lifting conjecture and the Embedding Conjecture of Abhyankar and Sathaye. Lifting seems to be “easier”.

5.2.5. Reduction to simple algebras. This subsection is devoted to finding test algebras.

Any prime algebra B satisfying a system of Capelli identities of order $n + 1$ (n minimal such) is said to have *rank* n . In this case, its operator algebra is PI. The localization of B is a simple algebra of dimension n over its centroid, which is a field. This is the famous *rank theorem* (see [156]).

5.2.5.1. Packing properties.

Definition 5.2.18. Let $\mathcal{M} = \{\mathfrak{M}_i : i \in I\}$ be an arbitrary set of varieties of algebras. We say that \mathcal{M} satisfies the *packing property*, if for any $n \in \mathbb{N}$ there exists a prime algebra A of rank n in some \mathfrak{M}_j such that any prime algebra in any \mathfrak{M}_i of rank n can be embedded into some central extension $K \otimes A$ of A .

\mathcal{M} satisfies the *finite packing property* if, for any finite set of prime algebras $A_j \in \mathfrak{M}_i$, there exists a prime algebra A in some \mathfrak{M}_k such that each A_j can be embedded into A .

The set of proper subvarieties of associative algebras satisfying a system of Capelli identities of some order k satisfies the packing property (because any simple associative algebra is a matrix algebra over field).

However, the varieties of alternative algebras satisfying a system of Capelli identities of order > 8 , or of Jordan algebras satisfying a system of Capelli identities of order > 27 , do not even satisfy the finite packing property. Indeed, the matrix algebra of order 2 and the Cayley-Dickson algebra cannot be embedded into a common prime alternative algebra. Similarly, \mathbb{H}_3 and the Jordan algebra of symmetric matrices cannot be embedded into a common Jordan prime algebra. (Both of these assertions follow easily by considering their PIs.)

It is not known whether or not the packing property holds for Engel algebras satisfying a system of Capelli identities; knowing the answer would enable us to resolve the JC, as will be seen below.

Theorem 5.2.19. *If the set of varieties of Engel algebras (of arbitrary fixed order) satisfying a system of Capelli identities of some order satisfies the packing property, then the Jacobian Conjecture has a positive solution.*

Theorem 5.2.20. *The set of varieties from the previous theorem satisfies the finite packing property.*

Most of the remainder of this section is devoted to the proof of these two theorems.

Problem. Using the packing property and deformations, give a reasonable analog of the JC in nonzero characteristic. (The naive approach using only the determinant of the Jacobian does not work.)

5.2.5.2. Construction of simple Yagzhev algebras. Using the Yagzhev correspondence and composition of elementary automorphisms it is possible to construct a new algebra of Engel type.

Theorem 5.2.21. *Let A be an algebra of Engel type. Then A can be embedded into a prime algebra of Engel type.*

Proof. Consider the mapping $F : V \rightarrow V$ (cf. (5.1.1)) given by

$$F : x_i \mapsto x_i + \sum_j \Psi_{ij}; \quad i = 1, \dots, n,$$

where the Ψ_{ij} are homogenous forms of degree j . Adjoining new indeterminates $\{t_i\}_{i=0}^n$, we put $F(t_i) = t_i$ for $i = 0, \dots, n$.

Now we take the transformation

$$G : t_0 \mapsto t_0, \quad x_i \mapsto x_i, \quad t_i \mapsto t_i + t_0 x_i^2 \quad \text{for } i = 1, \dots, n.$$

The composite $F \circ G$ has invertible Jacobian (and hence the corresponding algebra has Engel type) and can be expressed as follows:

$$F \circ G : x_i \mapsto x_i + \sum_j \Psi_{ij}, \quad t_0 \mapsto t_0, \quad t_i \mapsto t_i + t_0 x_i^2 \quad \text{for } i = 1, \dots, n.$$

It is easy to see that the corresponding algebra \hat{A} also satisfies the following properties:

- (a) \hat{A} contains A as a subalgebra (for $t_0 = 0$);

- (b) if A corresponds to a cubic homogenous mapping (and thus is Engel) then \widehat{A} also corresponds to a cubic homogenous mapping (and thus is Engel);
- (c) if some of the forms Ψ_{ij} are not zero, then A does not have nonzero ideals with product 0, and hence is prime (but its localization need not be simple!).

Any algebra A with operators can be embedded, using the previous construction, to a prime algebra with nonzero multiplication. The theorem is proved. \square

Embedding via the previous theorem preserves the cubic homogeneous case, but does not yet give us an embedding into a simple algebra of Engel type.

Theorem 5.2.22. *Any algebra A of Engel type can be embedded into a simple algebra of Engel type.*

Proof. We start from the following observation:

Lemma 5.2.23. *Suppose A is a finite dimensional algebra, equipped with a base e_1, \dots, e_n, e_{n+1} . If for any $1 \leq i, j \leq n+1$ there exist operators ω_{ij} in the signature $\Omega(A)$ such that $\omega_{ij}(e_i, \dots, e_i, e_{n+1}) = e_j$, with all other values on the base vectors being zero, then A is simple.*

This lemma implies:

Lemma 5.2.24. *Let F be a polynomial endomorphism of $\mathbb{C}[x_1, \dots, x_n; t_1, t_2]$, where*

$$F(x_i) = \sum_j \Psi_{ij}.$$

For notational convenience we put $x_{n+1} = t_1$ and $x_{n+2} = t_2$. Let $\{k_{ij}\}_{i=1,j}^s$ be a set of natural numbers such that

- *For any x_i there exists k_{ij} such that among all Ψ_{ij} there is exactly one term of degree k_{ij} , and it has the form $\Psi_{i,k_{ij}} = t_1 x_j^{k_{ij}-1}$.*
- *For t_2 and any x_i there exists k_{iq} such that among all Ψ_{ij} there is exactly one term of degree k_{iq} , and it has the form $\Psi_{n+2,k_{iq}} = t_1 x_j^{k_{iq}-1}$.*
- *For t_1 and any x_i there exists k_{iq} such that among all Ψ_{ij} there is exactly one term of degree k_{iq} , and it has the form $\Psi_{n+1,k_{iq}} = t_2 x_j^{k_{iq}-1}$.*

Then the corresponding algebra is simple.

Proof. Adjoin the term $t_\ell x_i^{k-1}$ to the x_i , for $\ell = 1, 2$. Let e_i be the base vector corresponding to x_i . Take the corresponding k_{ij} -ary operator

$$\omega : \omega(e_i, \dots, e_i, e_{n+\ell}) = e_j,$$

with all other products zero. Now we apply the previous lemma. \square

Remark 5.2.25. In order to be flexible with constructions via the Yagzhev correspondence, we are working in the general, not necessary cubic, case.

Now we can conclude the proof of Theorem 5.2.22. Let F be the mapping corresponding to the algebra A :

$$F : x_i \mapsto x_i + \sum_j \Psi_{ij}, \quad i = 1, \dots, n,$$

where Ψ_{ij} are forms of homogeneous degree j . Let us adjoin new indeterminates $\{t_1, t_2\}$ and put $F(t_i) = t_i$, for $i = 1, 2$.

We choose all $k_{\alpha,\beta} > \max(\deg(\Psi_{ij}))$ and assume that these numbers are sufficiently large. Then we consider the mappings

$$G_{k_{ij}} : x_i \mapsto x_i + x_j^{k_{ij}-1} t_1, \quad i \leq n; \quad t_1 \mapsto t_1; \quad t_2 \mapsto t_2; \quad x_s \mapsto x_s \text{ for } s \neq i.$$

$$G_{k_{i(n+2)}} : t_2 \mapsto x_i^{k_{ij}-1} t_1; \quad t_1 \mapsto t_1; \quad x_s \mapsto x_s \text{ for } 1 \leq s \leq n.$$

$$G_{k_{i(n+1)}} : t_1 \mapsto x_i^{k_{ij}-1} t_2; \quad t_2 \mapsto t_2; \quad x_s \mapsto x_s \text{ for } 1 \leq s \leq n.$$

These mappings are elementary automorphisms.

Consider the mapping $H = \circ_{k_{ij}} G_{k_{ij}} \circ F$, where the composite is taken in order of ascending $k_{\alpha\beta}$, and then with F . If the $k_{\alpha\beta}$ grow quickly enough, then the terms obtained in the previous step do not affect the lowest term obtained at the next step, and this term will be as described in the lemma. The theorem is proved. \square

Proof of Theorem 5.2.20. The direct sum of Engel type algebras is also of Engel type, and by Theorem 5.2.22 can be embedded into a simple algebra of Engel type. \square

The Yagzhev correspondence and algebraic extensions.

For notational simplicity, we consider a cubic homogeneous mapping

$$F : x_i \mapsto x_i + \Psi_{3i}(\mathbf{x}).$$

We shall construct the Yagzhev correspondence of an algebraic extension.

Consider the equation

$$t^s = \sum_{p=1}^s \lambda_p t^{s-p},$$

where the λ_p are formal parameters. If $m \geq s$, then for some λ_{pm} , which can be expressed as polynomials in $\{\lambda_p\}_{p=1}^{s-1}$, we have

$$t^m = \sum_{p=1}^s \lambda_{pm} t^{s-p}.$$

Let A be the algebra corresponding to the mapping F . Consider

$$A \otimes \mathbf{k}[\lambda_1, \dots, \lambda_s]$$

and its finite algebraic extension $\hat{A} = A \otimes \mathbf{k}[\lambda_1, \dots, \lambda_s, t]$. Now we take the mapping corresponding (via the Yagzhev correspondence) to the ground ring $R = \mathbf{k}[\lambda_1, \dots, \lambda_s]$ and algebra \hat{A} .

For $m = 1, \dots, s-1$, we define new formal indeterminates, denoted as $T^m x_i$. Namely, we put $T^0 x_i = x_i$ and for $m \geq s$, we identify $T^m x_i$ with

$$\sum_{p=1}^s \lambda_{pm} T^{s-p} x_i,$$

where $\{\lambda_p\}_{p=1}^{s-1}$ are formal parameters in the centroid of some extension $R \otimes A$. Now we extend the mapping F , by setting

$$F(T^m x_i) = T^m x_i + T^{3m} \Psi_{3i}(\mathbf{x}), \quad m = 1, \dots, s-1.$$

We get a natural mapping corresponding to the algebraic extension.

Now we can take more symbols T_j , $j = 1, \dots, s$, and equations

$$T_j^s = \sum_{p=1}^s \lambda_{pj} T_j^{s-p}$$

and a new set of indeterminates $x_{ijk} = T_j^k x_i$ for $j = 1, \dots, s$ and $i = 1, \dots, n$. Then we set

$$x_{ijm} = T_j^m x_i = \sum_{p=1}^s \lambda_{jpm} T_j^{s-p} x_i$$

and

$$F(x_{ijm}) = x_{ijm} + T_j^{3m} \Psi_{3i}(\mathbf{x}), \quad m = 1, \dots, s-1.$$

This yields an “algebraic extension” of A .

Deformations of algebraic extensions. Let $m = 2$. Let us introduce new indeterminates y_1, y_2 , put $F(y_i) = y_i$, $i = 1, 2$, and compose F with the automorphism

$$G : T_1^1 x_i \mapsto T_1^1 x_i + y_1 x_i, \quad T_1^1 x_i \mapsto T_2^1 x_i + y_1 x_i, \quad x_i \mapsto x_i, \quad i = 1, 2, \\ y_1 \mapsto y_1 + y_2^2 y_1, \quad y_2 \mapsto y_2.$$

(Note that the $T_1^1 x_i$ and $T_2^1 x_i$ are *new* indeterminates and not proportional to x_i !) Then compose G with the automorphism $H : y_2 \mapsto y_2 + y_1^2$, where H fixes the other indeterminates. Let us call the corresponding new algebra \hat{A} . It is easy to see that $\text{Var}(A) \neq \text{Var}(\hat{A})$.

Define an *identity of the pair* (A, B) , for $A \subseteq B$ to be a polynomial in two sets of indeterminates x_i, z_j that vanishes whenever the x_i are evaluated in A and z_j in B . The *variety of the pair* (A, B) is the class of pairs of algebras satisfying the identities of (A, B) .

Recall that by the rank theorem, any prime algebra A of rank n can be embedded into an n -dimensional simple algebra \hat{A} . We consider the variety of the pair (A, \hat{A}) .

Considerations of deformations yield the following:

Proposition 5.2.26. *Suppose for all simple n -dimensional pairs there exists a universal pair in which all of them can be embedded. Then the Jacobian Conjecture has a positive solution.*

We see the relation with the following assertion.

The Razmyslov–Kushkulei theorem [see [156]]. *Over an algebraically closed field, any two finite dimensional simple algebras satisfying the same identities are isomorphic.*

The difficulty in applying this theorem is that the identities may depend on parameters. Also, the natural generalization of the Razmyslov–Kushkulei theorem for a variety and subvariety does not hold: Even if $\text{Var}(B) \subset \text{Var}(A)$, where B and A are simple finite-dimensional algebras over some algebraically closed field, B need not be embeddable to A .

CONCLUSION

The quantization program constitutes a substantial and well designed approach to the Jacobian conjecture, as well as to various related topics in algebra and algebraic geometry. The recent developments presented in this review have been instrumental in our investigation of Kontsevich conjecture as well as the establishment of results of independent interest.

Furthermore, as can be seen from the discussion of the work of A.V. Yagzhev, there are substantial areas of the theory which require further development and which might, conceivably, hold the insights necessary for the resolution of the Jacobian conjecture.

While at present the quantization approach does not seem to be adequately developed for a successful attack on the Jacobian problem to happen (as evidenced by our discussion of Kontsevich conjecture), and the rather substantial critique of the general quantization and lifting philosophy (due to Orevkov and others) exists, further research and development of the theory is well advised.

BIBLIOGRAPHY

1. Abdesselam A. The Jacobian conjecture as a problem of perturbative quantum field theory// Ann. H. Poincaré. — 2003. — 4, № 2. — P. 199–215.
2. Abhyankar S., Moh T. Embedding of the line in the plane// J. Reine Angew. Math. — 1975. — 276. — P. 148–166.
3. Amitsur S. A. Algebras over infinite fields// Proc. Am. Math. Soc. — 1956. — 7. — P. 35–48.
4. Amitsur S. A. A general theory of radicals, III. Applications// Am. J. Math. — 1954. — 75. — P. 126–136.
5. Alev J., Le Bruyn L. Automorphisms of generic 2 by 2 matrices// in: Perspectives in Ring Theory. — Springer, 1988. — P. 69–83.
6. Amitsur A. S., Levitzki J. Minimal identities for algebras// Proc. Am. Math. Soc. — 1950. — 1. — P. 449–463.
7. Amitsur A. S., Levitzki J. Remarks on minimal identities for algebras// Proc. Am. Math. Soc. — 1951. — 2. — P. 320–327.

8. *Anick D. J.* Limits of tame automorphisms of $k[x_1, \dots, x_n]$ // J. Algebra. — 1983. — 82, № 2. — P. 459–468.
9. *Artamonov V. A.* Projective metabelian groups and Lie algebras// Izv. Math. — 1978. — 12, № 2. — C. 213–223.
10. *Artamonov V. A.* Projective modules over universal enveloping algebras// Math. USSR Izv. — 1985. — 25, № 3. — C. 429.
11. *Artamonov V. A.* Nilpotence, projectivity, decomposability// Sib. Math. J. — 1991. — 32, № 6. — C. 901–909.
12. *Artamonov V. A.* The quantum Serre problem// Russ. Math. Surv. — 1998. — 53, № 4. — C. 3–77.
13. *Artamonov V. A.* Automorphisms and derivations of quantum polynomials// in: Recent Advances in Lie Theory (*Bajo I., Sanmartin E.*, eds.). — Heldermann Verlag, 2002. — P. 109–120.
14. *Artamonov V. A.* Generalized derivations of quantum plane// J. Math. Sci. — 2005. — 131, № 5. — C. 5904–5918.
15. *Artamonov V. A.* *Quantum polynomials* in: Advances in Algebra and Combinatorics. — Singapore: World Scientific, 2008. — P. 19–34.
16. *Artin M.* Noncommutative Rings. — Preprint, 1999.
17. *Arzhantsev I., Kuyumzhiyan K., Zaidenberg M.* Infinite transitivity, finite generation, and Demazure roots// Adv. Math. — 2019. — 351. — P. 1–32.
18. *Asanuma T.* Non-linearizable algebraic k^* -actions on affine spaces. — Preprint, 1996.
19. *Backelin E.* Endomorphisms of quantized Weyl algebras// Lett. Math. Phys. — 2011. — 97, № 3. — P. 317–338.
20. *Bass H.* A non-triangular action of G_a on A^3 // J. Pure Appl. Algebra. — 1984. — 33, № 1. — P. 1–5.
21. *Bass H., Connell E. H., Wright D.* The Jacobian conjecture: reduction of degree and formal expansion of the inverse// Bull. Am. Math. Soc. — 1982. — 7, № 2. — P. 287–330.
22. *Bavula V. V.* A question of Rentschler and the Dixmier problem// Ann. Math. (2). — 2001. — 154, № 3. — P. 683–702.
23. *Bavula V. V.* Generalized Weyl algebras and diskew polynomial rings/ [arXiv:1612.08941 \[math.RA\]](#).
24. *Bavula V. V.* The group of automorphisms of the Lie algebra of derivations of a polynomial algebra// J. Alg. Appl. — 2017. — 16, № 5. — 1750088.
25. *Bavula V. V.* The groups of automorphisms of the Lie algebras of formally analytic vector fields with constant divergence// C. R. Math. — 2014. — 352, № 2. — P. 85–88.
26. *Bavula V. V.* The inversion formulae for automorphisms of Weyl algebras and polynomial algebras// J. Pure Appl. Algebra. — 2007. — 210. — P. 147–159.
27. *Bavula V. V.* The inversion formulae for automorphisms of polynomial algebras and rings of differential operators in prime characteristic// J. Pure Appl. Algebra. — 2008. — 212, № 10. — P. 2320–2337.
28. *Bavula V. V.* An analogue of the conjecture of Dixmier is true for the algebra of polynomial integro-differential operators// J. Algebra. — 2012. — 372. — P. 237–250.
29. *Bavula V. V.* Every monomorphism of the Lie algebra of unitriangular polynomial derivations is an automorphism// C. R. Acad. Sci. Paris. Ser. 1. — 2012. — 350, № 11–12. — P. 553–556.
30. *Bavula V. V.* The Jacobian conjecture $_{2n}$ implies the Dixmier problem $_n$ / [arXiv:math/0512250 \[math.RA\]](#).
31. *Beauville A., Colliot-Thelene J.-L., Sansuc J.-J., and Swinnerton-Dyer P.* Varietes stablement rationnelles non rationnelles// Ann. Math. — 1985. — 121. — P. 283–318.
32. *Bayen F., Flato M., Fronsdal C., Lichnerowicz A., Sternheimer D.* Deformation theory and quantization. I. Deformations of symplectic structures// Ann. Phys. — 1978. — 111, № 1. — P. 61–110.
33. *Belov A.* Linear recurrence equations on a tree// Math. Notes. — 2005. — 78, № 5. — C. 603–609.
34. *Belov A.* Local finite basis property and local representability of varieties of associative rings// Izv. Math. — 2010. — 74. — C. 1–126.
35. *Belov A., Bokut L., Rowen L., Yu J.-T.* The Jacobian conjecture, together with Specht and Burnside-type problems// in: Automorphisms in Birational and Affine Geometry. — Springer, 2014. — P. 249–285.
36. *Belov A., Makar-Limanov L., Yu J. T.* On the generalised cancellation conjecture// J. Algebra. — 2004. — 281. — P. 161–166.
37. *Belov A., Rowen L. H., Vishne U.* Structure of Zariski-closed algebras// Trans. Am. Math. Soc. — 2012. — 362. — P. 4695–4734.
38. *Belov-Kanel A., Yu J.-T.* On the lifting of the Nagata automorphism// Selecta Math. — 2011. — 17. — P. 935–945.

39. Kanel-Belov A., Berzins A., Lipyanski R. Automorphisms of the semigroup of endomorphisms of free associative algebras// Int. J. Algebra Comp. — 2007. — 17, № 5/6. — P. 923–939.
40. Belov-Kanel A., Elishev A. On planar algebraic curves and holonomic D -modules in positive characteristic// J. Algebra Appl. — 2016. — 15, № 8. — 1650155.
41. Belov-Kanel A., Kontsevich M. Automorphisms of the Weyl algebra// Lett. Math. Phys. — 2005. — 74, № 2. — P. 181–199.
42. Belov-Kanel A., Kontsevich M. The Jacobian conjecture is stably equivalent to the Dixmier conjecture// Moscow Math. J. — 2007. — 7, № 2. — C. 209–218.
43. Belov-Kanel A., Lipyanski R. Automorphisms of the endomorphism semigroup of a polynomial algebra// J. Algebra. — 2011. — 333, № 1. — P. 40–54.
44. Belov-Kanel A., Yu J.-T. Stable tameness of automorphisms of $F\langle x, y, z \rangle$ fixing z // Selecta Math. — 2012. — 18. — P. 799–802.
45. Bergman G. M. Centralizers in free associative algebras// Trans. Am. Math. Soc. — 1969. — 137. — P. 327–344.
46. Bergman G. M. The diamond lemma for ring theory// Adv. Math. — 1978. — 29, № 2. — P. 178–218.
47. Berson J., van den Essen A., Wright D. Stable tameness of two-dimensional polynomial automorphisms over a regular ring// Adv. Math. — 2012. — 230. — P. 2176–2197.
48. Birman J. An inverse function theorem for free groups// Proc. Am. Math. Soc. — 1973. — 41. — P. 634–638.
49. Bonnet P., Vénéreau S. Relations between the leading terms of a polynomial automorphism// J. Algebra. — 2009. — 322, № 2. — P. 579–599.
50. Berzins A. The group of automorphisms of semigroup of endomorphisms of free commutative and free associative algebras/ [arXiv: abs/math/0504015](#) [math.AG].
51. Białynicki-Birula A. Remarks on the action of an algebraic torus on k^n , I// Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. — 1966. — 14. — P. 177–181.
52. Białynicki-Birula A. Remarks on the action of an algebraic torus on k^n , II// Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. — 1967. — 15. — P. 123–125.
53. Białynicki-Birula A. Some theorems on actions of algebraic groups// Ann. Math. — 1973. — 98, № 3. — P. 480–497.
54. Bitoun T. The p -support of a holonomic D -module is lagrangian, for p large enough/ [arXiv: 1012.4081](#) [math.AG].
55. Bodnarchuk Yu. Every regular automorphism of the affine Cremona group is inner// J. Pure Appl. Algebra. — 2001. — 157. — P. 115–119.
56. Bokut L., Zelmanov E. Selected works of A. I. Shirshov. — Springer, 2009.
57. Bokut L. A. Embedding Lie algebras into algebraically closed Lie algebras// Algebra Logika. — 1962. — 1. — C. 47–53.
58. Bokut L. A. Embedding of algebras into algebraically closed algebras// Dokl. Akad. Nauk. — 1962. — 145, № 5. — C. 963–964.
59. Bokut L. A. Theorems of embedding in the theory of algebras// Colloq. Math. — 1966. — 14. — P. 349–353.
60. Brešar M., Procesi C., Špenko Š. Functional identities on matrices and the Cayley–Hamilton polynomial/ [arXiv: 1212.4597](#) [math.RA].
61. Campbell L. A. A condition for a polynomial map to be invertible// Math. Ann. — 1973. — 205, № 3. — P. 243–248.
62. Cohn P. M. Subalgebras of free associative algebras// Proc. London Math. Soc. — 1964. — 3, № 4. — P. 618–632.
63. Cohn P. M. Progress in free associative algebras// Isr. J. Math. — 1974. — 19, № 1-2. — P. 109–151.
64. Cohn P. M. A brief history of infinite-dimensional skew fields// Math. Sci. — 1992. — 17. — P. 1–14.
65. Cohn P. M. Free Rings and Their Relations. — Academic Press, 1985.
66. Czerniakiewicz A. J. Automorphisms of a free associative algebra of rank 2. I// Trans. Am. Math. Soc. — 1971. — 160. — P. 393–401.
67. Czerniakiewicz A. J. Automorphisms of a free associative algebra of rank 2. II// Trans. Am. Math. Soc. — 1972. — 171. — P. 309–315.

68. *Danielewski W.* On the cancellation problem and automorphism groups of affine algebraic varieties. — Warsaw: Preprint, 1989.
69. *De Bondt M., van den Essen A.* The Jacobian conjecture for symmetric Druzkowski mappings. — University of Nijmegen, 2004.
70. *De Bondt M., van den Essen A.* A reduction of the Jacobian conjecture to the symmetric case// *Proc. Am. Math. Soc.* — 2005. — 133, № 8. — P. 2201–2205.
71. *De Concini C., Procesi C.* A characteristic free approach to invariant theory// in: *Young Tableaux in Combinatorics, Invariant Theory, and Algebra.* — Elsevier, 1982. — P. 169–193.
72. *Déserti J.* Sur le groupe des automorphismes polynomiaux du plan affine// *J. Algebra.* — 2006. — 297. — P. 584–599.
73. *Dicks W.* Automorphisms of the free algebra of rank two// *Contemp. Math.* — 1985. — 43. — P. 63–68.
74. *Dicks W., Lewin J.* Jacobian conjecture for free associative algebras// *Commun. Algebra.* — 1982. — 10, № 12. — P. 1285–1306.
75. *Dixmier J.* Sur les algèbres de Weyl// *Bull. Soc. Math. France.* — 1968. — 96. — P. 209–242.
76. *Dodd C.* The p -cycle of holonomic D -modules and auto-equivalences of the Weyl algebra/ [arXiv:1510.05734 \[math.OA\]](#).
77. *Donkin S.* Invariants of several matrices// *Inv. Math.* — 1992. — 110, № 1. — P. 389–401.
78. *Donkin S.* Invariant functions on matrices// *Math. Proc. Cambridge Phil. Soc.* — 1993. — 113, № 1. — P. 23–43.
79. *Drensky V., Yu J.-T.* A cancellation conjecture for free associative algebras// *Proc. Am. Math. Soc.* — 2008. — 136, № 10. — P. 3391–3394.
80. *Drensky V., Yu J.-T.* The strong Anick conjecture// *Proc. Natl. Acad. Sci. U.S.A.* — 2006. — 103. — P. 4836–4840.
81. *Drensky V., Yu J.-T.* Coordinates and automorphisms of polynomial and free associative algebras of rank three// *Front. Math. China.* — 2007. — 2, № 1. — P. 13–46.
82. *Drensky V., Yu J.-T.* The strong Anick conjecture is true// *J. Eur. Math. Soc.* — 2007. — 9. — P. 659–679.
83. *Druzkowski L.* An effective approach to Keller’s Jacobian conjecture// *Math. Ann.* — 1983. — 264, № 3. — P. 303–313.
84. *Druzkowski L.* The Jacobian conjecture: symmetric reduction and solution in the symmetric cubic linear case// *Ann. Polon. Math.* — 2005. — 87, № 1. — P. 83–92.
85. *Druzkowski L. M.* New reduction in the Jacobian conjecture// in: *Effective Methods in Algebraic and Analytic Geometry.* — Kraków: Univ. Jagel. Acta Math., 2001. — P. 203–206.
86. *Elishev A.* Automorphisms of polynomial algebras, quantization and Kontsevich conjecture/ PhD Thesis — Moscow Institute of Physics and Technology, 2019.
87. *Elishev A., Kanel-Belov A., Razavinia F., Yu J.-T., Zhang W.* Noncommutative Białynicki-Birula theorem/ [arXiv:1808.04903 \[math.AG\]](#).
88. *Elishev A., Kanel-Belov A., Razavinia F., Yu J.-T., Zhang W.* Torus actions on free associative algebras, lifting and Białynicki-Birula type theorems/ [arXiv:1901.01385 \[math.AG\]](#).
89. *van den Bergh M.* On involutivity of p -support// *Int. Math. Res. Not.* — 2015. — 15. — P. 6295–6304.
90. *van den Essen A.* The amazing image conjecture/ [arXiv:1006.5801 \[math.AG\]](#).
91. *van den Essen A., de Bondt M.* Recent progress on the Jacobian conjecture// *Ann. Polon. Math.* — 2005. — 87. — P. 1–11.
92. *van den Essen A., de Bondt M.* The Jacobian conjecture for symmetric Druzkowski mappings// *Ann. Polon. Math.* — 2005. — 86, № 1. — P. 43–46.
93. *van den Essen A., Wright D., Zhao W.* On the image conjecture// *J. Algebra.* — 2011. — 340. — P. 211–224.
94. *Fox R. H.* Free differential calculus, I. Derivation in the free group ring// *Ann. Math. (2).* — 1953. — 57. — P. 547–560.
95. *Gizatullin M. Kh., Danilov V. I.* Automorphisms of affine surfaces, I// *Izv. Math.* — 1975. — 9, № 3. — C. 493–534.
96. *Gizatullin M. Kh., Danilov V. I.* Automorphisms of affine surfaces, II// *Izv. Math.* — 1977. — 11, № 1. — C. 51–98.
97. *Gorni G., Zampieri G.* Yagzhev polynomial mappings: on the structure of the Taylor expansion of their local inverse// *Polon. Math.* — 1996. — 64. — P. 285–290.

98. Fedosov B. A simple geometrical construction of deformation quantization// J. Differ. Geom. — 1994. — 40, № 2. — P. 213–238.
99. Frayne T., Morel A. C., Scott D. S. Reduced direct products// J. Symb. Logic.. — 31, № 3. — P. 1966.
100. Fulton W., Harris J. Representation Theory. A First Course. — Springer-Verlag, 1991.
101. Furter J.-P., Kraft H. On the geometry of the automorphism groups of affine varieties/ [arXiv:1809.04175 \[math.AG\]](#).
102. Gutwirth A. The action of an algebraic torus on the affine plane// Trans. Am. Math. Soc. — 1962. — 105, № 3. — P. 407–414.
103. Jung H. W. E. Über ganze birationale Transformationen der Ebene// J. Reine Angew. Math. — 1942. — 184. — P. 161–174.
104. Kaliman S., Koras M., Makar-Limanov L., Russell P. C^* -actions on C^3 are linearizable// Electron. Res. Announc. Am. Math. Soc. — 1997. — 3. — P. 63–71.
105. Kaliman S., Zaidenberg M. Families of affine planes: the existence of a cylinder// Michigan Math. J. — 2001. — 49. — P. 353–367.
106. Kuroda S. Shestakov–Umirbaev reductions and Nagata’s conjecture on a polynomial automorphism// Tôhoku Math. J. — 2010. — 62. — P. 75–115.
107. Kuzmin E., Shestakov I. P. Nonassociative structures// Itogi Nauki Tekhn. Sovr. Probl. Mat. Fundam. Napr. — 1990. — 57. — C. 179–266.
108. Karas’ M. Multidegrees of tame automorphisms of C^n // Dissert. Math. — 2011. — 477.
109. Khoroshkin A., Piontkovski D. On generating series of finitely presented operads/ [arXiv:1202.5170 \[math.QA\]](#).
110. Kambayashi T. Pro-affine algebras, Ind-affine groups and the Jacobian problem// J. Algebra. — 1996. — 185, № 2. — P. 481–501.
111. Kambayashi T. Some basic results on pro-affine algebras and Ind-affine schemes// Osaka J. Math. — 2003. — 40, № 3. — P. 621–638.
112. Kambayashi T., Russell P. On linearizing algebraic torus actions// J. Pure Appl. Algebra. — 1982. — 23, № 3. — P. 243–250.
113. Kanel-Belov A., Borisenko V., Latysev V. Monomial algebras// J. Math. Sci. — 1997. — 87, № 3. — C. 3463–3575.
114. Kanel-Belov A., Elishev A. On planar algebraic curves and holonomic \mathcal{D} -modules in positive characteristic/ [arXiv:1412.6836 \[math.AG\]](#).
115. Kanel-Belov A., Elishev A., Yu J.-T. Independence of the B-KK isomorphism of infinite prime/ [arXiv:1512.06533 \[math.AG\]](#).
116. Kanel-Belov A., Elishev A., Yu J.-T. Augmented polynomial symplectomorphisms and quantization/// [arXiv:1812.02859 \[math.AG\]](#).
117. Kanel-Belov A., Grigoriev S., Elishev A., Yu J.-T., Zhang W. Lifting of polynomial symplectomorphisms and deformation quantization// Commun. Algebra. — 2018. — 46, № 9. — P. 3926–3938.
118. Kanel-Belov A., Malev S., Rowen L. The images of noncommutative polynomials evaluated on 2×2 matrices// Proc. Am. Math. Soc. — 2012. — 140. — P. 465–478.
119. Kanel-Belov A., Malev S., Rowen L. The images of multilinear polynomials evaluated on 3×3 matrices// Proc. Am. Math. Soc. — 2016. — 144. — P. 7–19.
120. Kanel-Belov A., Razavinia F., Zhang W. Bergman’s centralizer theorem and quantization// Commun. Algebra. — 2018. — 46, № 5. — P. 2123–2129.
121. Kanel-Belov A., Razavinia F., Zhang W. Centralizers in free associative algebras and generic matrices/ [arXiv:1812.03307 \[math.RA\]](#).
122. Kanel-Belov A., Rowen L. H., Vishne U. Full exposition of Specht’s problem// Serdica Math. J. — 2012. — 38. — P. 313–370.
123. Kanel-Belov A., Yu J.-T., Elishev A. On the augmentation topology of automorphism groups of affine spaces and algebras// Int. J. Algebra Comput. * — 2018. — 28, № 08. — P. 1449–1485.
124. Keller B. Notes for an Introduction to Kontsevich’s Quantization Theorem, 2003.
125. Keller O. H. Ganze Cremona Transformationen// Monatsh. Math. Phys. — 1939. — 47, № 1. — P. 299–306.
126. Kolesnikov P. S. The Makar-Limanov algebraically closed skew field// Algebra Logic. — 2000. — 39, № 6. — C. 378–395.

127. *Kolesnikov P. S.* Different definitions of algebraically closed skew fields// *Algebra Logic.* — 2001. — 40, № 4. — C. 219–230.
128. *Kontsevich M.* Deformation quantization of Poisson manifolds// *Lett. Math. Phys.* — 2003. — 66, № 3. — P. 157–216.
129. *Kontsevich M.* Holonomic D -modules and positive characteristic// *Jpn. J. Math.* — 2009. — 4, № 1. — P. 1–25.
130. *Koras M., Russell P.* C^* -actions on C^3 : The smooth locus of the quotient is not of hyperbolic type// *J. Alg. Geom.* — 1999. — 8, № 4. — P. 603–694.
131. *Kovalenko S., Perepechko A., Zaidenberg M.* On automorphism groups of affine surfaces// in: *Algebraic Varieties and Automorphism Groups.* — Math. Soc. Jpn., 2017. — P. 207–286.
132. *Kraft H., Regeta A.* Automorphisms of the Lie algebra of vector fields// *J. Eur. Math. Soc.* — 2017. — 19, № 5. — P. 1577–1588.
133. *Kraft H., Stampfli I.* On automorphisms of the affine Cremona group// *Ann. Inst. Fourier.* — 2013. — 63, № 3. — P. 1137–1148.
134. *Kulikov V. S.* Generalized and local Jacobian problems// *Izv. Math.* — 1993. — 41, № 2. — C. 351–365.
135. *Kulikov V. S.* The Jacobian conjecture and nilpotent maps// *J. Math. Sci.* — 2001. — 106, № 5. — C. 3312–3319.
136. *Levy R., Loustaunau P., Shapiro J.* The prime spectrum of an infinite product of copies of Z // *Fundam. Math.* — 1991. — 138. — P. 155–164.
137. *Li Y.-C., Yu J.-T.* Degree estimate for subalgebras// *J. Algebra.* — 2012. — 362. — P. 92–98.
138. *Gaiotto D., Witten E.* Probing quantization via branes/ [arXiv:2107.12251 \[hep-th\]](#).
139. *Lothaire M.* *Combinatorics on Words.* — Cambridge Univ. Press, 1997.
140. *Makar-Limanov L.* A new proof of the Abhyankar–Moh–Suzuki theorem/ [arXiv:1212.0163 \[math.AC\]](#).
141. *Makar-Limanov L.* Automorphisms of a free algebra with two generators// *Funct. Anal. Appl.* — 1970. — 4, № 3. — C. 262–264.
142. *Makar-Limanov L.* On automorphisms of Weyl algebra// *Bull. Soc. Math. France.* — 1984. — 112. — P. 359–363.
143. *Makar-Limanov L., Yu J.-T.* Degree estimate for subalgebras generated by two elements// *J. Eur. Math. Soc.* — 2008. — 10. — P. 533–541.
144. *Makar-Limanov L.* Algebraically closed skew fields// *J. Algebra.* — 1985. — 93, № 1. — P. 117–135.
145. *Makar-Limanov L., Turusbekova U., Umirbaev U.* Automorphisms and derivations of free Poisson algebras in two variables// *J. Algebra.* — 2009. — 322, № 9. — P. 3318–3330.
146. *Markl M., Shnider S., Stasheff J.* *Operads in Algebra, Topology, and Physics.* — Providence, Rhode Island: Am. Math. Soc., 2002.
147. *Miyanishi M., Sugie T.* Affine surfaces containing cylinderlike open sets// *J. Math. Kyoto Univ.* — 1980. — 20. — P. 11–42.
148. *Nagata M.* *On the automorphism group of $k[x, y]$.* — Tokyo: Kinokuniya, 1972.
149. *Nielsen J.* Die Isomorphismen der allgemeinen, unendlichen Gruppen mit zwei Erzeugenden// *Math. Ann.* — 1918. — 78. — P. 385–397.
150. *Nielsen J.* Die Isomorphismengruppe der freien Gruppen// *Math. Ann.* — 1924. — 91. — P. 169–209.
151. *Ol’shanskij A. Yu.* Groups of bounded period with subgroups of prime order// *Algebra and Logic.* — 1983. — 21. — C. 369–418.
152. *Peretz R.* Constructing polynomial mappings using non-commutative algebras// in: *Affine Algebraic Geometry.* — Providence, Rhode Island: Am. Math. Soc., 2005. — P. 197–232.
153. *Piontkovski D.* *Operads versus Varieties: a dictionary of universal algebra.* — Preprint, 2011.
154. *Piontkovski D.* On Kurosh problem in varieties of algebras// *J. Math. Sci.* — 2009. — 163, № 6. — C. 743–750.
155. *Razmyslov Yu. P.* Algebras satisfying identity relations of Capelli type// *Izv. Akad. Nauk SSSR. Ser. Mat.* — 1981. — 45. — C. 143–166, 240.
156. *Razmyslov Yu. P.* *Identities of Algebras and Their Representations.* — Providence, Rhode Island: Am. Math. Soc., 1994.
157. *Razmyslov Yu. P., Zubrilin K. A.* Nilpotency of obstacles for the representability of algebras that satisfy Capelli identities, and representations of finite type// *Russ. Math. Surveys* — 1993. — 48. — C. 183–184.

158. *Reutenauer C.* Applications of a noncommutative Jacobian matrix// J. Pure Appl. Algebra. — 1992. — 77. — P. 634–638.
159. *Rowen L. H.* Graduate Algebra: Noncommutative View. — Providence, Rhode Island: Am. Math. Soc., 2008.
160. *Moh T.-T.* On the global Jacobian conjecture for polynomials of degree less than 100. — Preprint, 1983.
161. *Moh T.-T.* On the Jacobian conjecture and the configurations of roots// J. Reine Angew. Math. — 1983. — 340. — P. 140–212.
162. *Moyal J. E.* Quantum mechanics as a statistical theory// Math. Proc. Cambridge Philos. Soc. — 1949. — 45, № 1. — P. 99–124.
163. *Orevkov S. Yu.* The commutant of the fundamental group of the complement of a plane algebraic curve// Russ. Math. Surv. — 1990. — 45, № 1. — C. 221–222.
164. *Orevkov S. Yu.* An example in connection with the Jacobian conjecture// Math. Notes. — 1990. — 47, № 1. — C. 82–88.
165. *Orevkov S. Yu.* The fundamental group of the complement of a plane algebraic curve// Sb. Math. — 1990. — 65, № 1. — C. 267–267.
166. *Plotkin B.* Varieties of algebras and algebraic varieties// Israel J. Math. — 1996. — 96, № 2. — P. 511–522.
167. *Plotkin B.* Algebras with the same (algebraic) geometry/ [arXiv:math/0210194](#) [math.GM].
168. *Popov V. L.* Around the Abhyankar-Sathaye conjecture/ [arXiv:1409.6330](#) [math.AG].
169. *Procesi C.* Rings with Polynomial Identities. — Marcel Dekker, 1973.
170. *Procesi C.* The invariant theory of $n \times n$ matrices// Adv. Math. — 1976. — 19, № 3. — P. 306–381.
171. *Razar M.* Polynomial maps with constant Jacobian// Israel J. Math. — 1979. — 32, № 2-3. — P. 97–106.
172. *Robinson A.* Non-Standard Analysis. — Princeton Univ. Press, 2016.
173. *Rosset S.* A new proof of the Amitsur–Levitzki identity// Israel J. Math. — 1976. — 23, № 2. — P. 187–188.
174. *Rowen L. H.* Graduate Algebra: Noncommutative View. — Providence, Rhode Island: Am. Math. Soc., 2008.
175. *Schofield A. H.* Representations of Rings over Skew Fields. — Cambridge: Cambridge Univ. Press, 1985.
176. *Schwarz G.* Exotic algebraic group actions// C. R. Acad. Sci. Paris — 1989. — 309. — P. 89–94.
177. *Shafarevich I. R.* On some infinite-dimensional groups, II// Izv. Ross. Akad. Nauk. Ser. Mat. — 1981. — 45, № 1. — C. 214–226.
178. *Sharifi Y.* Centralizers in Associative Algebras/ Ph.D. thesis, 2013.
179. *Shestakov I. P.* Finite-dimensional algebras with a nil basis// Algebra Logika. — 1971. — 10. — C. 87–99.
180. *Shestakov I. P., Umirbaev U. U.* Degree estimate and two-generated subalgebras of rings of polynomials// J. Am. Math. Soc. — 2004. — 17. — P. 181–196.
181. *Shestakov I., Umirbaev U.* The Nagata automorphism is wild// Proc. Natl. Acad. Sci. — 2003. — 100, № 22. — P. 12561–12563.
182. *Shestakov I., Umirbaev U.* Poisson brackets and two-generated subalgebras of rings of polynomials// J. Am. Math. Soc. — 2004. — 17, № 1. — P. 181–196.
183. *Shestakov I. P., Umirbaev U. U.* The tame and the wild automorphisms of polynomial rings in three variables// J. Am. Math. Soc. — 2004. — 17. — P. 197–220.
184. *Umirbaev U., Shestakov I.* Subalgebras and automorphisms of polynomial rings// Dokl. Ross. Akad. Nauk — 2002. — 386, № 6. — C. 745–748.
185. *Shpilrain V.* On generators of L/R^2 Lie algebras// Proc. Am. Math. Soc. — 1993. — 119. — P. 1039–1043.
186. *Singer D.* On Catalan trees and the Jacobian conjecture// Electron. J. Combin. — 2001. — 8, № 1. — 2.
187. *Shpilrain V., Yu J.-T.* Affine varieties with equivalent cylinders// J. Algebra. — 2002. — 251, № 1. — P. 295–307.
188. *Shpilrain V., Yu J.-T.* Factor algebras of free algebras: on a problem of G. Bergman// Bull. London Math. Soc. — 2003. — 35. — P. 706–710.
189. *Suzuki M.* Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébrique de l'espace C^2 // J. Math. Soc. Jpn. — 1974. — 26. — P. 241–257.
190. *Tsuchimoto Y.* Preliminaries on Dixmier conjecture Mem. Fac. Sci. Kochi Univ. Ser. A. Math. — 2003. — 24. — P. 43–59.
191. *Tsuchimoto Y.* Endomorphisms of Weyl algebra and p -curvatures// Osaka J. Math. — 2005. — 42, № 2. — P. 435–452.

192. *Tsuchimoto Y.* Auslander regularity of norm based extensions of Weyl algebra// [arXiv:1402.7153 \[math.AG\]](#).
193. *Umirbaev U.* On the extension of automorphisms of polynomial rings// *Sib. Math. J.* — 1995. — 36, № 4. — C. 787–791.
194. *Umirbaev U. U.* On Jacobian matrices of Lie algebras// В КН.: Proc. 6 All-Union Conf. on Varieties of Algebraic Systems. — Magnitogorsk, 1990. — C. 32–33.
195. *Umirbaev U. U.* Shreer varieties of algebras// *Algebra Logic.* — 1994. — 33. — C. 180–193.
196. *Umirbaev U. U.* Tame and wild automorphisms of polynomial algebras and free associative algebras. — Preprint MPIM 2004–108..
197. *Umirbaev U.* The Anick automorphism of free associative algebras// *J. Reine Angew. Math.* — 2007. — 605. — P. 165–178.
198. *Umirbaev U. U.* Defining relations of the tame automorphism group of polynomial algebras in three variables// *J. Reine Angew. Math.* — 2006. — 600. — P. 203–235.
199. *Umirbaev U. U.* Defining relations for automorphism groups of free algebras// *J. Algebra.* — 2007. — 314. — P. 209–225.
200. *Umirbaev U. U., Yu J.-T.* The strong Nagata conjecture// *Proc. Natl. Acad. Sci. U.S.A.* — 2004. — 101. — P. 4352–4355.
201. *Urech C., Zimmermann S.* Continuous automorphisms of Cremona groups/ [arXiv:1909.11050 \[math.AG\]](#).
202. *van den Essen A.* Polynomial Automorphisms and the Jacobian Conjecture. — Birkhäuser, 2012.
203. *van der Kulk W.* On polynomial rings in two variables// *Nieuw Arch. Wisk.* (3) — 1953. — 1. — P. 33–41.
204. *Vitushkin A. G.* A criterion for the representability of a chain of σ -processes by a composition of triangular chains// *Math. Notes* — 1999. — 65, № 5-6. — C. 539–547.
205. *Vitushkin A. G.* On the homology of a ramified covering over C^2 // *Math. Notes.* — 1998. — 64, № 5. — C. 726–731.
206. *Vitushkin A. G.* Evaluation of the Jacobian of a rational transformation of C^2 and some applications// *Math. Notes* — 1999. — 66, № 2. — C. 245–249.
207. *Wedderburn J. H. M.* Note on algebras// *Ann. Math.* — 1937. — 38. — P. 854–856.
208. *Wright D.* The Jacobian conjecture as a problem in combinatorics/ [arXiv:math/0511214 \[math.CO\]](#).
209. *Wright D.* The Jacobian conjecture: Ideal membership questions and recent advances// *Contemp. Math.* — 2005. — 369. — P. 261–276.
210. *Yagzhev A. V.* Finiteness of the set of conservative polynomials of a given degree// *Math. Notes.* — 1987. — 41, № 2. — C. 86–88.
211. *Yagzhev A. V.* Nilpotency of extensions of an abelian group by an abelian group// *Math. Notes.* — 1988. — 43, № 3-4. — C. 244–245.
212. *Yagzhev A. V.* Locally nilpotent subgroups of the holomorph of an abelian group// *Mat. Zametki* — 1989. — 46, № 6. — C. 118.
213. *Yagzhev A. V.* A sufficient condition for the algebraicity of an automorphism of a group// *Algebra Logic.* — 1989. — 28, № 1. — C. 83–85.
214. *Yagzhev A. V.* The generators of the group of tame automorphisms of an algebra of polynomials// *Sib. Mat. Zh.* — 1977. — 18, № 1. — P. 222–225.
215. *Wang S.* A Jacobian criterion for separability// *J. Algebra.* — 1980. — 65, № 2. — P. 453–494.
216. *Wright D.* On the Jacobian conjecture// *Ill. J. Math.* — 1981. — 25, № 3. — P. 423–440.
217. *Yagzhev A. V.* Invertibility of endomorphisms of free associative algebras// *Math. Notes.* — 1991. — 49, № 3–4. — C. 426–430.
218. *Yagzhev A. V.* Endomorphisms of free algebras// *Sib. Math. J.* — 1980. — 21, № 1. — C. 133–141.
219. *Yagzhev A. V.* On the algorithmic problem of recognizing automorphisms among endomorphisms of free associative algebras of finite rank// *Sib. Math. J.* — 1980. — 21, № 1. — C. 142–146.
220. *Yagzhev A. V.* Keller's problem// *Sib. Math. J.* — 1980. — 21, № 5. — C. 747–754.
221. *A. V. Yagzhev* Engel algebras satisfying Capelli identities// В КН.: Proceedings of Shafarevich Seminar. — Moscow: Steklov Math. Inst., 2000. — C. 83–88 (in Russian).
222. *A. V. Yagzhev* Endomorphisms of polynomial rings and free algebras of different varieties// В КН.: Proceedings of Shafarevich Seminar. — Moscow: Steklov Math. Inst., 2000. — C. 15–47 (in Russian).
223. *Yagzhev A. V.* Invertibility criteria of a polynomial mapping. — Unpublished (in Russian).

224. *Zaks A.* Dedekind subrings of $K[x_1, \dots, x_n]$ are rings of polynomials// Israel J. Math. — 1971. — 9. — P. 285–289.
225. *Zelmanov E.* On the nilpotence of nilalgebras// Lect. Notes Math. — 1988. — 1352. — P. 227–240.
226. *Zhao W.* New proofs for the Abhyankar–Gurjar inversion formula and the equivalence of the Jacobian conjecture and the vanishing conjecture// Proc. Am. Math. Soc. — 2011. — 139. — P. 3141–3154.
227. *Zhao W.* Mathieu subspaces of associative algebras// J. Algebra. — 2012. — 350. — P. 245–272.
228. *Zhevlakov K. A., Slin'ko A. M., Shestakov I. P., Shirshov A. I.* Nearly Associative Rings. — Moscow: Nauka, 1978 (in Russian).
229. *Zubrilin K. A.* Algebras that satisfy the Capelli identities// Sb. Math. — 1995. — 186, № 3. — C. 359–370.
230. *Zubrilin K. A.* On the class of nilpotence of obstruction for the representability of algebras satisfying Capelli identities// Fundam. Prikl. Mat. — 1995. — 1, № 2. — C. 409–430.
231. *Zubrilin K. A.* On the Baer ideal in algebras that satisfy the Capelli identities// Sb. Math. — 1998. — 189. — C. 1809–1818.
232. *Zaidenberg M. G.* On exotic algebraic structures on affine spaces// in: Geometric Complex Analysis. — World Scientific, 1996. — P. 691–714.
233. *Zhang W.* Alternative proof of Bergman's centralizer theorem by quantization/ Master thesis — Bar-Ilan University, 2017.
234. *Zhang W.* Polynomial automorphisms and deformation quantization/ Ph.D. thesis — Bar-Ilan University, 2019.
235. *Zubkov A. N.* Matrix invariants over an infinite field of finite characteristic// Sib. Math. J. — 1993. — 34, № 6. — C. 1059–1065.
236. *Zubkov A. N.* A generalization of the Razmyslov–Procesi theorem// Algebra Logic. — 1996. — 35, № 4. — C. 241–254.

Елишев Андрей Михайлович

Московский физико-технический институт (национальный исследовательский университет)

E-mail: ame1511@mail.ru

Канель-Белов Алексей Яковлевич

Московский физико-технический институт (национальный исследовательский университет)

E-mail: kanelster@gmail.com

Razavinia Farrokh

Московский физико-технический институт (национальный исследовательский университет)

E-mail: farrokh.razavinia@gmail.com

Jie-Tai Yu

Шэньчженьский университет, Шэньчжень, Китайская народная республика

E-mail: yujt@hkust.hk

Wenchao Zhang

Школа математики и статистики, Университет Хуэйчжоу, Китайская народная республика

E-mail: zhangwc@hzu.edu.cn