УДК 517.929.7
DOI: 10.20310/1810-0198-2016-21-1-76-81

# POISSON PROBLEM FOR A LINEAR FUNCTIONAL DIFFERENTIAL EQUATION

© S. Labovskiy, M. F. Getimane

The solvability, existence and positiveness of the Green function of the Poisson problem

$$-\Delta u-\int\limits_{\Omega}(u(y)-u(x))\,r(x,dy)=\rho f,\quad u\big|_{\Gamma(\Omega)}=0$$

are showed. The spectral properties of corresponding eigenvalue problem are considered. Here $\Omega$ is an open set in $\mathbb{R}^n$, and $\Gamma(\Omega)$ is the boundary of the $\Omega$. For almost all $x\in\Omega$, $r(x,\cdot)$ is a measure satisfying certain symmetry condition. The function $\rho$ is a positive weight. This problem has a clear mechanical interpretation.
*Key words:* poisson problem; discreteness of spectrum; Green function positivity; functional differential equation.

## 1. The Poisson problem

### 1.1. The problem

Let $\Omega$ be an open set in $\mathbb{R}^n$, and $\Gamma(\Omega)$ be the boundary of the $\Omega$. We study the Poisson problem

$$-\Delta u-\int\limits_{\Omega}(u(y)-u(x))\,r(x,dy)=\rho f, \tag{1}$$

$$u\big|_{\Gamma(\Omega)}=0 \tag{2}$$

where $x=(x_1,\dots,x_n)$, $\Delta u=u''_{x_1x_1}+\dots+u''_{x_nx_n}$. For almost all $x\in\Omega$, the function $r(x,\cdot)$ is a measure satisfying certain symmetry condition. The function $\rho$ is a positive weight.

The boundary value problem (1),(2) may describe behavior of an mechanical object. Due to this mechanical interpretation, we can predict some of the properties of the boundary value problem. Problems with a similar approach were considered in [1, 2, 3].

The scheme of method is as following. Represent the problem (1),(2) in the form

$$\mathcal{L}u=f,$$

where

$$\mathcal{L}u=-\frac{1}{\rho}\left(\Delta u+\int\limits_{\Omega}(u(y)-u(x))\,r(x,dy)\right). \tag{3}$$

Using variational method we show unique solvability of the problem. Multiplying by $v$, and integrating, we obtain an equation of the form

$$[u,v]=\int_{\Omega}fv\,dx, \tag{4}$$

where $[u,v]$ is a bilinear form. Therefore, we start with the bilinear form[4]

$$[u,v] \stackrel{\text{def}}{=} \int\limits_{\Omega} u'_x v'_x \, dx + \frac{1}{2} \int\limits_{\Omega\times\Omega} (u(y)-u(x))(v(y)-v(x))\, \xi(dx \times dy), \tag{5}$$

$dx = dx_1 \cdots dx_n$ . Then we show that the equation (4) with relation to $u$ has a unique solution in a space. The operator $\mathcal{L}$ will be constructed automatically.

## 1.2. Assumptions and notation

Let $\Omega \subset \mathbb{R}^n$ be a nonempty *bounded open* set, $\Gamma(\Omega)$ be the boundary of the $\Omega$ , and $X = \overline{\Omega}$ be the closure of $\Omega$ . For a real function $u = u(x)$ defined on $\Omega$ and having derivative of first order,

$$u'_x = \left(u'_{x_1}, \ldots, u'_{x_n}\right),$$

where $x = (x_1, \ldots, x_n)$ . For two such functions $u$ and $v$ ,

$$u'_x v'_x = u'_{x_1} v'_{x_1} + \cdots + u'_{x_n} v'_{x_n}.$$

### 1.2.1. The form (5) we use under following assumptions

Let $\mathcal{M}$ be the set of all Lebesgue measurable subsets in $X = \overline{\Omega}$ . Let the function $r\colon X \times \mathcal{M} \to R$ satisfy two conditions: for almost all $x \in X$ , the function $r(x,\cdot)$ is a measure on $(X, \mathcal{M})$ , for any $e \in \mathcal{M}$ , $r(\cdot, e)$ is measurable on $X$ .

The set functiom $\xi$ defined by the equality

$$\xi(E) = \int_X r(x, E_x)\, dx, \;\; E_x = \{y\colon (x,y) \in E\} \tag{6}$$

is a measure. Assume that $\xi$ is symmetric, i.e.

$$\xi(e_1 \times e_2) = \xi(e_2 \times e_1), \; \forall e_1, e_2 \in \mathcal{M}. \tag{7}$$

The form $[u,v]$ defined by (5) we will use in the Sobolev space $W_0^{1,2}(\Omega)$ .

D e f i n i t i o n 1. Let $W$ be the vector subspace of all elements from $W_0^{1,2}(\Omega)$ satisfying $[u,u] < \infty$ .

Define the operator $T\colon W \to L_2(\Omega)$ by the equality $Tu(x) = u(x)$ , $x \in \Omega$ . The operator $T$ is continuous.

### 1.2.2. $L_2(X,\mu)$

Let $\rho(x)$ , $x \in X$ , be a positive measurable weight (density). The measure $\mu(S) = \int_S \rho(x) dx$ we may imagine as the mass of a measurable part $S \subset X$ . Let

$$(f,g) = \int\limits_{\Omega} f(x) g(x) \rho(x)\, dx. \tag{8}$$

Let $L_2(X,\mu)$ (or simply $L_2(X)$ , or $L_2(\Omega)$ ) be the set of all $\mu$-measurable functions on $X$ with finite integral $\int\limits_{\Omega} f(x)^2 \rho(x)\, dx$ . The $L_2(X,\mu)$ is Hilbert space.

[4]the notation $\xi(dx \times dy)$ is equivalent to $d\xi$

### 1.3. Representation of the operator $\mathcal{L}$

So, our bilinear form $[u,v]$ is inner product in the space $W$. The image $T(W)$ is dense in $L_2(X)$. Note first that the equation $[u,v]=(f,Tv)$ has the form

$$\int\limits_{\Omega} u'_x v'_x\,dx + \frac{1}{2}\int\limits_{\Omega\times\Omega}(u(y)-u(x))(v(y)-v(x))\xi(dx\times dy)$$

$$= \int_{\Omega} f(x)v(x)\rho(x)\,dx, \quad \forall v\in W. \quad (9)$$

For any $f\in L_2(\Omega)$ it has a unique solution $u=T^*f\in W$.

Now we confirm the representation (3) of operator $\mathcal{L}$.

T h e o r e m 1. $W_0^{2,2}(\Omega)\subset D(\mathcal{L})$ and in $W_0^{2,2}(\Omega)$ operator $\mathcal{L}$ has representation (3).

P r o o f. For any $u\in C_0^\infty(\Omega)$ or $u\in W_0^{2,2}(\Omega)$ and $v\in W$ integrating by parts one can easily obtain the identity

$$\int\limits_{\Omega} u'_x v'_x\,dx = -\int\limits_{\Omega}\Delta u\cdot v\,dx.$$

Hence, if $\Delta u=g$, then

$$\int\limits_{\Omega} u'_x v'_x\,dx = -\int\limits_{\Omega} g\cdot v\,dx, \quad \forall v\in W. \quad (10)$$

Note that $C_0^\infty\subset D(\mathcal{L})$. We take equation (10) as *definition of operator* $\Delta$ on space $D(\mathcal{L})$ in a weak sense.

For any $u,v\in W$

$$\frac{1}{2}\int\limits_{\Omega\times\Omega}(u(y)-u(x))(v(y)-v(x))\,d\xi = -\int\limits_{\Omega} dx\,v(x)\int\limits_{\Omega}(u(y)-u(x))\,r(x,dy).$$

From (9) and by definition of the $\Delta$

$$\Delta u = -\int\limits_{\Omega}(u(y)-u(x))\,r(x,dy)-\rho f. \quad (11)$$

Thus, the operator $\mathcal{L}$ has representation

$$\mathcal{L}u = -\frac{1}{\rho}\left(\Delta u+\int\limits_{\Omega}(u(y)-u(x))\,r(x,dy)\right).$$

### 1.4. Eigenvalue problem and spectrum of $\mathcal{L}$

T h e o r e m 2. Let $\Omega$ satisfy the *cone condition* [4, Paragraph 4.6]. The eigenvalue problem

$$-\Delta u-\int\limits_{\Omega}(u(y)-u(x))\,r(x,dy)=\lambda\rho u, \quad (12)$$

$$u\big|_{\Gamma(\Omega)}=0 \quad (13)$$

has in $W$ a system of nontrivial solutions $u_n(x)$ corresponding to positive eigenvalues $\lambda_n$. This system forms an orthogonal basis in the space $W_0^{1,2}$.

R e m a r k. Expression $\Delta u$ is defined in the weak sense (10).

R e m a r k. Each element $u\in W_0^{1,2}$ satisfies the boundary condition $u\big|_{\Gamma(\Omega)}=0$ in the following sense. Since limit value of any $u\in C_0^\infty(\Omega)$ on $\Gamma(\Omega)$ is zero, it may be taken as value of $u\in W_0^{1,2}$ on the $\Gamma(\Omega)$.

## 2. Positivity of solutions

L e m m a Suppose $f \geq 0$ and the solution $u(x)$ of the problem (1),(2) is in $C_0^\infty(\Omega)$. Then $u(x) \geq 0$ in $\Omega$.

P r o o f. Suppose $u(x_0) < 0$. Let $u(x_0) < h < 0$ and $E = \{x \in \Omega \colon u(x) < h\}$. Since

$$\int_E dx \int_E (u(y) - u(x)) r(x, dy) = \int_{E \times E} (u(y) - u(x)) d\xi = 0$$

we have

$$\int_E dx \int_X (u(y) - u(x)) r(x, dy) = \int_E dx \int_{X \setminus E} (u(y) - u(x)) r(x, dy) \geq 0.$$

Now

$$\int_E \Delta u \, dx = \int_{\Gamma(E)} \frac{\partial u}{\partial \nu} \, ds,$$

where in the right side[5] is integral on boundary $\Gamma(E)$ of $E$ of the derivative of $u$ along on outer normal $\nu$ of $u$. This integral is positive for some $h \in (u(x_0), 0)$. So

$$\int_E \left( \Delta u + \int_X (u(y) - u(x)) r(x, dy) + f \right) dx > 0.$$

This contradiction proves the lemma.

L e m m a 1. Suppose $f \geq 0$ and $u(x)$ is the solution of the problem (1),(2). Then $u(x) \geq 0$ in $\Omega$.

P r o o f. Since $W$ is closure of $C_0^\infty(\Omega)$ the $u$ can be represented as limit $u = \lim\limits_{n \to \infty} u_n$, where $u_n \in C_0^\infty$. Let $f_n = \mathcal{L} u_n$, $f_n = f_n^+ - f_n^-$ where $f_n^+ = (f_n + |f_n|)/2$. Let $D_n = \{x \colon f_n^+(x) = 0\}$. Since

$$\int_\Omega (f - f_n)^2 \, dx \geq \int_{D_n} (f - f_n)^2 \, dx = \int_{D_n} (f + f_n^-)^2 \, dx \geq \int_{D_n} (f_n^-)^2 \, dx$$

the sequence $f_n^-$ tends to zero in $L_2(\Omega)$. Since $u_n = u_n^+ - u_n^-$, where $u_n^\pm = T^* f_n^\pm$,

$$u = \lim u_n = \lim u_n^+.$$

This limit is a nonnegative function because by lemma 1 $u_n^+ \geq 0$.

T h e o r e m 3. Suppose $f \geq \not\equiv 0$ and $u(x)$ is the solution of the problem (1),(2). Then $u(x) > 0$ in $\Omega$.

P r o o f. From (1)

$$-\Delta u + p(x) u = \int_\Omega u(s) r(x, ds) + \rho(x) f(x),$$

where $p(x) = r(x, \Omega)$. By lemma 1 the right side is nonnegative. Since $p \geq 0$ the solution of this equation with boundary condition (2) is positive in $\Omega$.

C o r o l l a r y. The minimal eigenvalue $\lambda_0$ of the problem (12),(13) is positive and simple. It associated with a positive in $\Omega$ eigenfunction.

T h e o r e m The Green operator $G = T^*$ has the integral representation

$$T^* f(x) = \int_\Omega G(x, s) f(s) \rho(s) \, ds. \tag{14}$$

[5]this follows from the formula $\int_E v \Delta u \, dx = -\int_E v_x u_x \, dx + \int_{\Gamma(E)} v \frac{\partial u}{\partial \nu} \, ds$

P r o o f. Let $u = T^* f$ and $x_0 = (x_0^1, \dots, x_0^n) \in \Omega$ is a fixed point. Then $u \mapsto u(x_0)$ is a linear functional defined on $C_0^\infty$. It is bounded as

$$u(x_0) = -\int_{x_0^1}^{s} \frac{\partial u}{\partial x^1}\, dx^1$$

where $x_0 = (s, x_0^2, \dots, x_0^n)$ is the nearest point of $\Gamma(\Omega)$ on the line $x^i = x_0^i$, $i = 2, \dots, n$, $x^1 \geq x_0^1$, and

$$|u(x_0)|^2 \leq \int_{x_0^1}^{s} \left(\frac{\partial u}{\partial x^1}\right)^2 dx^1 \cdot \int_{x_0^1}^{s} dx^1 \leq C[u,u].$$

Let $\varphi_{x_0}$ be the extension of this functional on all space $W$.

This functional has the representation $\varphi_{x_0}(u) = [u, g_{x_0}]$, $g_{x_0} \in W$,

$$\varphi_{x_0}(u) = [T^* f, g_{x_0}] = (f, T g_{x_0}) = \int_\Omega g_{x_0}(s) f(s) \rho(s)\, ds.$$

Let $G(x,s) = g_x(s)$. For $u \in C_0^\infty(\Omega)$ the formula (14) gives exact value for any $x \in \Omega$. If $u \in W$, it is $\lim u_n$, $u_n \in C_0^\infty(\Omega)$, and the (14) is fulfilled almost everywhere on $\Omega$.

## REFERENCES

1. *Labovskiĭ S.M.* On the Sturm-Liouville problem for a linear singular functional-differential equation // Russ. Math., 1996. V. 40, № 11. P. 50–56. (English. Russian original. Translation from Izv. Vyssh. Uchebn. Zaved., Mat. 1996, № 11 (414), 48-53).

2. *Labovskii S.* Little vibrations of an abstract mechanical system and corresponding eigenvalue problem // Functional Differential Equations, 1999. V. 6, № 1-2. P. 155–167.

3. *Labovskiy S.* On spectral problem and positive solutions of a linear singular functional differential equation // Functional Differential Equations, 2013. V. 20, № 3-4. P. 179–200.

4. *Adams R.A. and Fournier J.* Sobolev Spaces // Elsevier, 2003.

Received 12 January 2016.

Labovskiy Sergei Mikhailovich, Plekhanov Russian University of Economics, Moscow, the Russian Federation, Candidate of Physics and Mathematics, Associate Professor of the Higher Mathematics Department, e-mail: labovski@gmail.com

Getimane Mário Frengue, Instituto Superior de Transportes e Comunicações, Maputo, Mozambique, Associate Professor, Mathematics Department, e-mail: mgetimane@isutc.transcom.co.mz

UDC 517.929.7
DOI: 10.20310/1810-0198-2016-21-1-76-81

# ЗАДАЧА ПУАССОНА ДЛЯ ЛИНЕЙНОГО ФУНКЦИОНАЛЬНО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ

© С. М. Лабовский, М. Ф. Жетимане

Показаны разрешимость, существование и положительность функции Грина задачи Пуассона

$$-\Delta u - \int\limits_{\Omega} (u(y) - u(x))\, r(x, dy) = \rho f, \quad u\big|_{\Gamma(\Omega)} = 0$$

Рассмотрены спектральные свойства соответствующей задачи на собственные значения. Здесь $\Omega$ – открытое множество в $\mathbb{R}^N$ и $\Gamma(\Omega)$ является границей $\Omega$. Для почти всех $x \in \Omega$, $r(x, \cdot)$ является мерой, удовлетворяющей определенному условию симметрии. Функция $\rho$ – положительный вес. Задача имеет ясную механическую интерпретацию.

*Key words:* задача Пуассона; дискретность спектра; положительность функции Грина; функционально-дифференциальное уравнение.

## СПИСОК ЛИТЕРАТУРЫ

1. *Labovskiĭ S.M.* On the Sturm-Liouville problem for a linear singular functional-differential equation // Russ. Math., 1996. V. 40, № 11. P. 50–56. (English. Russian original. Translation from Izv. Vyssh. Uchebn. Zaved., Mat. 1996, № 11 (414), 48-53).
2. *Labovskii S.* Little vibrations of an abstract mechanical system and corresponding eigenvalue problem // Functional Differential Equations, 1999. V. 6, № 1-2. P. 155–167.
3. *Labovskiy S.* On spectral problem and positive solutions of a linear singular functional differential equation // Functional Differential Equations, 2013. V. 20, № 3-4. P. 179–200.
4. *Adams R.A. and Fournier J.* Sobolev Spaces // Elsevier, 2003.

Поступила в редакцию 12 января 2016 г.

Лабовский Сергей Михайлович, Российский экономический университет им. Г. В. Плеханова, г. Москва, Российская Федерация, кандидат физико-математических наук, доцент кафедры высшей математики, e-mail: labovski@gmail.com

Жетимане Марио Френге, Институт транспорта и сообщений, Мапуто, Мозамбик, доцент кафедры математики, e-mail: mgetimane@isutc.transcom.co.mz