ISSN 2686-9667. Вестник российских университетов. Математика

Том 30, № 152 2025

SCIENTIFIC ARTICLE



https://doi.org/10.20310/2686-9667-2025-30-152-322-337

# On the control problem for a pseudo-parabolic equation with involution in a bounded domain

**Farrukh N. DEKHKONOV**[1] **, Batirkhan Kh. TURMETOV**[2,3]

[1] Namangan State University

316 Uychi St., Namangan 1600136, Uzbekistan

[2] Khoja Akhmet Yassawi International Kazakh-Turkish University

29 Bekzat Sattarhanov St., Turkistan 161200, Kazakhstan

[3] Alfraganus University

2a Yukori Karakamish Str., Tashkent 100190, Uzbekistan

**Abstract.** This paper considers a control problem for a pseudo-parabolic equation with an involution operator in a bounded domain. A generalized solution to the corresponding initial boundary value problem is obtained. By introducing an additional integral condition, the control problem is reduced to a Volterra integral equation of the first kind. To show that the integral equation has a solution, some estimates are obtained for the kernel of this integral equation. The existence of a solution to the integral equation is shown using the Laplace transform method and the admissibility of the control function is proved.

**Keywords:** pseudo-parabolic equation, Volterra integral equation, admissible control, initial boundary value problem, Laplace transform, weight function

**Acknowledgements:** The research was supported by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant no. AP23488086).

**Mathematics Subject Classification:** 35K70, 35K35.

**For citation:** Dekhkonov F.N., Turmetov B.Kh. On the control problem for a pseudo-parabolic equation with involution in a bounded domain. *Vestnik rossiyskikh universitetov. Matematika = Russian Universities Reports. Mathematics*, **30**:152 (2025), 322–337.
https://doi.org/10.20310/2686-9667-2025-30-152-322-337

НАУЧНАЯ СТАТЬЯ



https://doi.org/10.20310/2686-9667-2025-30-152-322-337
УДК 517.977

# О задаче управления для псевдопараболического уравнения с инволюцией в ограниченной области

**Фаррух Нуриддин угли ДЕХКОНОВ**[1] **, Батирхан Худайбергенович ТУРМЕТОВ**[2,3]
[1] Наманганский государственный университет
160136, Узбекистан, г. Наманган, ул. Уйчи, 316
[2] Международный казахско-турецкий университет имени Ходжи Ахмеда Ясави
161200, Казахстан, г. Туркестан, ул. Бекзата Саттарханова, 29
[3] Университет Альфраганус
100190, Узбекистан, г. Ташкент, ул. Юкори Каракамыш, 2а

**Аннотация.** В данной работе рассматривается задача управления для псевдопараболического уравнения с оператором инволюции в ограниченной области. Получено обобщенное решение соответствующей начально-краевой задачи. Путем введения дополнительного интегрального условия задача управления сведена к интегральному уравнению Вольтерра первого рода. Для того чтобы показать, что интегральное уравнение имеет решение, получены некоторые оценки для ядра этого интегрального уравнения. С помощью метода преобразования Лапласа показано существование решения интегрального уравнения и доказана допустимость функции управления.

**Ключевые слова:** псевдопараболическое уравнение, интегральное уравнение Вольтерра, допустимое управление, начально-краевая задача, преобразование Лапласа, весовая функция

**Благодарности:** Исследование выполнено при финансовой поддержке Комитета науки Министерства науки и высшего образования Республики Казахстан (грант № AP23488086).

**Для цитирования:** *Дехконов Ф.Н., Турметов Б.Х.* О задаче управления для псевдопараболического уравнения с инволюцией в ограниченной области // Вестник российских университетов. Математика. 2025. Т. 30. № 152. С. 322–337.
https://doi.org/10.20310/2686-9667-2025-30-152-322-337

## Introduction

Pseudo-parabolic equations arise in various applied contexts, including fluid flow, heat transfer, and radiative diffusion [1, 2]. These models extend classical parabolic equations by capturing memory and inertial effects, allowing for higher-order diffusion modeling.

In recent years, growing interest in control theory and applied mathematics has spurred extensive research on control problems for pseudo-parabolic equations. Foundational contributions include the study of control and its comparison with classical parabolic models [3], and investigations on stability, uniqueness, and existence of solutions for various pseudo-parabolic systems [4]. Point control for parabolic and pseudoparabolic models is discussed in detail in [5].

The foundations of control problems for parabolic equations were laid by Friedman [6], with significant progress on controllability achieved by Fattorini and Russell [7]. Egorov [8] extended the theory to infinite-dimensional settings by generalizing Pontryagin's maximum principle and proving a bang-bang type result. Comprehensive overviews of optimal control for PDEs are presented in the monographs by Fursikov [9] and Lions [10].

The bang-bang principle in time-optimal boundary control problems was rigorously examined in [11], where it was shown that such controls are optimal for driving the system to any prescribed temperature profile. Initial studies on time-optimal control for heat conduction in bounded $n$-dimensional domains were presented in [12], where estimates for the minimal time required to reach a prescribed average temperature were established. The boundary control problem for the heat equation in a two-dimensional domain was investigated in [13], where the control function was applied to steer the temperature distribution toward a desired state, and its admissibility was established via geometric bounding techniques. Boundary control problems for heat transfer equations in [14] were further studied in the two-dimensional domain, focusing on the thermal regulation of inhomogeneous media. In [15], null-controllability of two degenerate heat equations with nonlocal spatial terms was established using Carleman estimates adapted to inhomogeneous coupled models.

In [16,17], boundary control problems for pseudo-parabolic equations were further explored, where the construction of a suitable control function was achieved to ensure the heating of an inhomogeneous thin plate up to a prescribed temperature distribution.

In recent years, increasing attention has been paid to mixed problems for parabolic-type equations involving involution. Various inverse problems for such equations have been analyzed, for instance, in [18,19]. The study [20] addresses a boundary value problem for the heat equation with an involution term in a one-dimensional spatial domain. A number of boundary value problems involving parabolic equations with involution have also been investigated in [21,22]. In [23], a fourth-order parabolic equation with an involution is studied under appropriate boundary conditions.

The resolution of inverse problems for a nonlocal analogue of a fourth-order parabolic equation in a multidimensional parallelepiped domain is presented in [24]. In [25], an inverse problem for a fractional-order parabolic equation with a nonlocal biharmonic operator is thoroughly analyzed in a two-dimensional domain. The study [26] addresses inverse problems for the heat equation with involution under Dirichlet, Neumann, periodic, and antiperiodic boundary conditions, establishing existence and uniqueness results for each case. In addition, the works [27, 28] investigated boundary control problems for heat equations with involution, where the Laplace transform was used to find control functions ensuring the desired average temperature.

In this paper, we address a boundary control problem for a pseudo-parabolic equation involving an involution operator, with the principal aim of proving the existence of an admissible control function. By applying the method of separation of variables, the control problem is reduced to a Volterra integral equation of the first kind (see Section 2). Although establishing the solvability of such integral equations is known to be a nontrivial task, we employ the Laplace transform technique to construct a solution, thereby confirming the existence of the desired control. In Section 3, we derive key estimates for the integral kernel, which play a central role in the analysis. Section 4 is devoted to establishing bounds that ensure the admissibility of the control function.

## 1. Statement of problem. Main result

Consider a bounded domain $\Omega \subset \mathbb{R}^n$ in the form of an open rectangular parallelepiped defined by

$$\Omega := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 < x_i < p_i, \quad i = 1, \dots, n\},$$

where each $p_i > 0$ denotes the length of the domain along the $x_i$-axis. The boundary of $\Omega$ is denoted by $\partial\Omega$.

Consider the family of mappings $S_i : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$S_i x = (x_1, \dots, x_{i-1}, p_i - x_i, x_{i+1}, \dots, x_n),$$

where each $S_i$ reflects the $i$-th coordinate of the vector $x$ with respect to the midpoint of the interval $(0, p_i)$. It is straightforward to verify that each $S_i$ is an involution, i. e., $S_i^2 = I$, where $I$ denotes the identity transformation on $\mathbb{R}^n$.

Let us consider all possible products of mappings $S_i$, i. e., $S_{ji} = S_j S_j$, or $S_{jik} = S_j S_j S_k, \dots$. The total number of such mappings, taking into account the identity mapping $S_0 x = x$, is equal to $2^n$. To number such mappings, we will use the binary number system, namely, if $0 \le j < 2^n$ in the binary number system, the representation $j \equiv (j_n \dots j_1)_2 = j_1 + 2j_2 + \dots + 2^{n-1} j_n$, where $j_k$ takes one of the values $0$ or $1$. Therefore, introduction the vector $j = (j_1, \dots, j_n)$, we can consider mappings of the type $S_j \equiv S_1^{j_1} \dots S_n^{j_n}$ corresponding to the index $j$.

Using these mappings, we define a nonlocal operator $\mathcal{L}_n$ acting on a function $U(x)$ by

$$\mathcal{L}_n U(x) = \sum_{j=0}^{2^n-1} a_j \, \Delta U(S_j x),$$

where $a_0, \dots, a_{2^n-1}$ are given real coefficients, and $\Delta$ denotes the standard Laplace operator. We refer to $\mathcal{L}_n$ as a nonlocal analogue of the Laplacian due to its dependence on spatial reflections.

It is known that the operator $\mathcal{L}_n$ is defined as follows for $n = 2$ :

$$\mathcal{L}_2 U(x_1, x_2) = a_0 \Delta U(x_1, x_2) + a_1 \Delta U(p_1 - x_1, x_2) + a_2 \Delta U(x_1, p_2 - x_2) + a_3 \Delta U(p_1 - x_1, p_2 - x_2).$$

In this work, we investigate the following pseudo-parabolic equation with involution:

$$u_t(x,t) - \mathcal{L}_n \, u(x,t) - \mathcal{L}_n \, u_t(x,t) = h(x) \, \nu(t), \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

with the homogeneous Dirichlet boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.2}$$

and the initial condition

$$u(x,0)=0, \quad x\in\overline{\Omega}. \tag{1.3}$$

Here, $h(x)$ is a given function, and $\nu(t)$ is a control function. We say that function $\nu\in L_2(\mathbb{R}_+)$ is an *admissible control* if it satisfies the pointwise constraint $|\nu(t)|\le 1$ for almost every $t\ge 0$.

We now consider the following spectral problem:

$$\mathcal{L}_n w(x)+\lambda\, w(x)=0, \quad x\in\Omega, \tag{1.4}$$

with the homogeneous Dirichlet boundary condition

$$w(x)=0, \quad x\in\partial\Omega. \tag{1.5}$$

In the special case when $a_0=1$ and $a_j=0$ for all $j=1,\dots,n$, the operator $\mathcal{L}_n$ coincides with the classical Laplace operator, and the spectral problem (1.4)–(1.5) reduces to the standard Dirichlet problem for the Laplacian.

To determine the eigenvalues and eigenfunctions of problem (1.4)–(1.5), we first recall the corresponding classical spectral problem:

$$\Delta\vartheta(x)+\mu\,\vartheta(x)=0, \quad x\in\Omega, \quad \vartheta(x)=0, \quad x\in\partial\Omega. \tag{1.6}$$

It is well known that the eigenfunctions of problem (1.6) are given by

$$\vartheta_{m_1,\dots,m_n}(x)=C_n\prod_{k=1}^{n}\sin\frac{m_k\pi x_k}{p_k}, \quad m_k\in\mathbb{N}, \tag{1.7}$$

where $C_n = 2^{n/2}\prod_{k=1}^{n}\frac{1}{\sqrt{p_k}}$ is the normalization constant ensuring that the functions are orthonormal in $L_2(\Omega)$ (see, e. g., [29, p. 331]).

The system of functions defined by (1.7) forms a complete orthonormal basis in the space $L_2(\Omega)$. The corresponding eigenvalues are given by

$$\mu_{m_1,\dots,m_n}=\pi^2\sum_{k=1}^{n}\frac{m_k^2}{p_k^2}, \quad m_k\in\mathbb{N}. \tag{1.8}$$

Thus, the eigenfunctions of problem (1.4)–(1.5) are given by (see [30])

$$w_{m_1,\dots,m_n}(x)=\vartheta_{m_1,\dots,m_n}(x)=C_n\prod_{k=1}^{n}\sin\frac{m_k\pi x_k}{p_k},$$

and the corresponding eigenvalues take the form

$$\lambda_{m_1,\dots,m_n}=\theta_{m_1,\dots,m_n}\pi^2\sum_{k=1}^{n}\frac{m_k^2}{p_k^2},$$

where

$$\theta_{m_1,\dots,m_n}=\sum_{j=0}^{2^n-1}a_j\,(-1)^{|j|+j_1m_1+j_2m_2+\dots+j_nm_n}, \quad m_k\in\mathbb{N}.$$

Here, for the index $j$ represented in binary as $j=(j_n\dots j_1)_2$ with $j_k\in\{0,1\}$, $|j|=j_1+j_2+\dots+j_n$ denotes the Hamming weight of the binary vector.

Since the system of functions $\{\vartheta_{m_1,\ldots,m_n}(x)\}$ is complete in $L_2(\Omega)$, it follows that the eigenfunctions of problem (1.4)–(1.5) coincide with those of the classical Dirichlet Laplacian whenever $\theta_{m_1,\ldots,m_n} \neq 0$. In the subsequent analysis, we assume that $\theta_{m_1,\ldots,m_n} > 0$ for all $m_k \in \mathbb{N}$ and $k = 1,\ldots,n$.

As a consequence of this completeness, and since $w_{m_1,\ldots,m_n}(x) = \vartheta_{m_1,\ldots,m_n}(x)$, the following lemma holds.

**Lemma 1.1** (see [30]). *The system of functions* $w_{m_1,\ldots,m_n}(x)$, $m_k \in \mathbb{N}$, $k = \overline{1,n}$ *are orthonormal and complete in space* $L_2(\Omega)$.

We define the class of admissible weight functions $\rho(x)$ as

$$\mathcal{A}(\Omega) := \left\{ \rho \in \mathring{W}_2^1(\Omega) \;\middle|\; \rho(x) \geq 0, \quad \frac{\partial}{\partial x_k}\rho(x) \leq 0, \quad \int_\Omega \rho(x)\,dx = 1, \quad k = 1,\ldots,n \right\},$$

where $\mathring{W}_2^1(\Omega)$ denotes the Sobolev space of functions that vanish on the boundary of $\Omega$ in the trace sense.

Assume that the Fourier coefficients of the given function $h(x)$ with respect to the eigenfunctions $\{w_{m_1,\ldots,m_n}(x)\}$ satisfy the non-negativity condition

$$h_{m_1,\ldots,m_n} \geq 0, \quad m_k \in \mathbb{N}, \quad k = 1,\ldots,n, \tag{1.9}$$

where the coefficients are defined by

$$h_{m_1,\ldots,m_n} = \int_0^{p_1} \cdots \int_0^{p_n} h(x_1,\ldots,x_n)\, w_{m_1,\ldots,m_n}(x_1,\ldots,x_n)\, dx_1,\ldots,dx_n.$$

Consider the weight function

$$\rho(x) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \rho_{m_1,\ldots,m_n}\, w_{m_1,\ldots,m_n}(x), \quad x \in \Omega,$$

where $\rho_{m_1,\ldots,m_n}$ are the Fourier coefficients of the function $\rho(x)$, i. e.

$$\rho_{m_1,\ldots,m_n} = \int_\Omega \rho(x)\, w_{m_1,\ldots,m_n}(x)\, dx \quad m_k \in \mathbb{N}, \quad k = 1,\ldots,n.$$

The present study is devoted to the analysis of the following thermal control problem:

**Control Problem.** *For given a measurable function* $\phi\colon \mathbb{R}_+ \to \mathbb{R}$, *find an admissible control function* $\nu \in L_2(\mathbb{R}_+)$ *such that the solution* $u(x,t)$ *of the mixed problem* (1.1)–(1.3) *satisfies the integral condition*

$$\int_\Omega \rho(x)\, u(x,t)\, dx = \phi(t), \quad \forall t \geq 0. \tag{1.10}$$

Physically, the function $\phi(t)$ corresponds to the desired average temperature within the domain $\Omega$. The primary aim of this study is to construct an admissible control function that ensures the temperature of the system evolves in such a way that the prescribed average temperature $\phi(t)$ is attained for all $t \geq 0$.

For any constant $M > 0$, we define the class $W(M)$ as the set of functions $\phi \in W_2^2(\mathbb{R})$ satisfying the conditions

$$\|\phi\|_{W_2^2(\mathbb{R}_+)} \leq M, \quad \phi(t) = 0 \quad \text{for} \quad t \leq 0.$$

We now state the main result of this paper.

**Theorem 1.1.** *There exists constant $M > 0$ such that for any function $\phi \in W(M)$, the solution $\nu(t)$ of the equation (1.10) exists and satisfies condition $|\nu(t)| \le 1$.*

The proof of this theorem will be developed through a sequence of auxiliary results and estimates in the following sections.

## 2. Main integral equation

In this section, we demonstrate how the control problem governed by the pseudo-parabolic equation can be reduced to an equivalent Volterra integral equation of the first kind. This reduction enables us to apply analytical techniques for integral equations in order to investigate the existence, uniqueness, and admissibility of the control function.

Let $B$ be a Banach space and $T > 0$ a fixed constant. We denote by $C([0,T] \to B)$ the Banach space of all continuous mappings $u\colon [0,T] \to B$ endowed with the norm $\|u\|_T := \sup_{t\in[0,T]} \|u(t)\|_B$. This space is itself a Banach space with respect to the given norm.

By $\mathring{W}_2^1(\Omega)$, we denote the subspace of the Sobolev space $W_2^1(\Omega)$ consisting of functions with zero trace on $\partial\Omega$. Note that due to the closure $\mathring{W}_2^1(\Omega)$ the sum of a series of functions from $\mathring{W}_2^1(\Omega)$, converging in metric $W_2^1(\Omega)$ also belongs to $\mathring{W}_2^1(\Omega)$.

D e f i n i t i o n 2.1. Assume that $h \in L_2(\Omega)$ is given, and let $\nu \in L_2([0,T])$. A function $u(x,t)$ is called a generalized solution of problem (1.1)–(1.3) if:

1. $u \in C([0,T] \to \mathring{W}_2^1(\Omega))$;
2. $u(x,0) = 0$ holds for almost every $x \in \Omega$;
3. For every test function $\chi \in \mathring{W}_2^1(\Omega)$ and for all $t \in [0,T]$, the identity holds:

$$\int_\Omega u_t(x,t)\,\chi(x)dx + \sum_{j=0}^{2^n-1} a_j \int_\Omega \sum_{i=1}^{n} (-1)^{j_i} \frac{\partial}{\partial x_i} u(S_1^{j_1}\dots S_i^{j_i}\dots S_n^{j_n}x,t) \frac{\partial}{\partial x_i}\chi(x)\,dx$$
$$+ \sum_{j=0}^{2^n-1} a_j \int_\Omega \sum_{i=1}^{n} (-1)^{j_i} \frac{\partial}{\partial x_i} u_t(S_1^{j_1}\dots S_i^{j_i}\dots S_n^{j_n}x,t) \frac{\partial}{\partial x_i}\chi(x)\,dx = \nu(t)\int_\Omega h(x)\,\chi(x)\,dx.$$

The class $C([0,T] \to \mathring{W}_2^1(\Omega))$ is a subset of the class $W_2^{1,0}(\Omega_T)$, which was taken into consideration in monograph [31] for defining a solution to the problem homogeneous boundary conditions (refer to the corresponding uniqueness theorem in Chapter III, Theorem 3.2, pp. 173-176), where $\Omega_T = \Omega \times (0,T)$. Accordingly, the generalized solution mentioned above is likewise a generalized solution in the sense of [31]. However, a solution from the class $C([0,T] \to \mathring{W}_2^1(\Omega))$ continually relies on $t \in [0,T]$ in the metric $L_2(\Omega)$, in contrast to a solution from the class $W_2^{1,0}(\Omega_T)$, which is guaranteed to have a trace for practically everywhere $t \in [0,T]$.

**Lemma 2.1.** *Let $h \in L_2(\Omega)$ and $\nu \in L_2(\mathbb{R}_+)$. Assume that $\theta_{m_1,\dots,m_n} > 0$ for all $m_k \in \mathbb{N}$, $k = \overline{1,n}$. Then the function*

$$u(x,t) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{h_{m_1,\dots,m_n}}{1+\lambda_{m_1,\dots,m_n}} \left(\int_0^t \nu(s) e^{-q_{m_1,\dots,m_n}(t-s)} ds\right) w_{m_1,\dots,m_n}(x), \tag{2.1}$$

*is the unique solution of the problem (1.1)–(1.3) in the class $C([0,T] \to \mathring{W}_2^1(\Omega))$, where $h_{m_1,\dots,m_n}$ are the Fourier coefficients of the function $h(x)$, $q_{m_1,\dots,m_n} = \frac{\lambda_{m_1,\dots,m_n}}{1+\lambda_{m_1,\dots,m_n}}$ and $\lambda_{m_1,\dots,m_n} = \theta_{m_1,\dots,m_n}\,\mu_{m_1,\dots,m_n} > 0$.*

P r o o f. By using the suggested Fourier series, we will demonstrate that the function $u(x,t)$ is a member of the class $C([0,T] \to \mathring{W}_2^1(\Omega))$. This function's gradient, measured with regard to $x \in \Omega$, may be shown to depend continuously on $t \in [0,T]$ on the space $L_2(\Omega)$ norm. According to Parseval's equality, the norm of this gradient is equal to

$$\begin{aligned}
\|\nabla u(\cdot,t)\|^2_{L_2(\Omega)} &= \int_\Omega |\nabla u(x,t)|^2 dx \\
&= \int_\Omega \Big| \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty \frac{h_{m_1,\dots,m_n}}{1+\lambda_{m_1,\dots,m_n}} \Big( \int_0^t \nu(s) e^{-q_{m_1,\dots,m_n}(t-s)} ds \Big) \nabla w_{m_1,\dots,m_n}(x) \Big|^2 dx \\
&\le \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty \Big| \frac{h_{m_1,\dots,m_n}}{1+\lambda_{m_1,\dots,m_n}} \int_0^t \nu(s) e^{-q_{m_1,\dots,m_n}(t-s)} ds \Big|^2 \int_\Omega |\nabla w_{m_1,\dots,m_n}(x)|^2 dx \\
&= \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty \Big| \frac{h_{m_1,\dots,m_n}}{1+\lambda_{m_1,\dots,m_n}} \int_0^t \nu(s) e^{-q_{m_1,\dots,m_n}(t-s)} ds \Big|^2 \mu_{m_1,\dots,m_n} \\
&\le \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty \frac{|h_{m_1,\dots,m_n}|^2}{(1+\lambda_{m_1,\dots,m_n})^2} \mu_{m_1,\dots,m_n} \Big( \int_0^t |\nu(s)| e^{-q_{m_1,\dots,m_n}(t-s)} ds \Big)^2 \\
&\le \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty \frac{|h_{m_1,\dots,m_n}|^2}{(1+\lambda_{m_1,\dots,m_n})^2} \mu_{m_1,\dots,m_n} \Big( \int_0^t e^{-2q_{m_1,\dots,m_n}(t-s)} ds \Big) \Big( \int_0^t |\nu(s)|^2 ds \Big) \\
&= \|\nu\|^2_{L_2(0,T)} \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty \frac{|h_{m_1,\dots,m_n}|^2}{(1+\lambda_{m_1,\dots,m_n})^2} \mu_{m_1,\dots,m_n} \Big( \frac{1-e^{-2q_{m_1,\dots,m_n}t}}{2\, q_{m_1,\dots,m_n}} \Big) \\
&= \|\nu\|^2_{L_2(0,T)} \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty \frac{|h_{m_1,\dots,m_n}|^2}{1+\lambda_{m_1,\dots,m_n}} \Big( \frac{1-e^{-2q_{m_1,\dots,m_n}t}}{2\, \theta_{m_1,\dots,m_n}} \Big) \\
&\le C\, \|\nu\|^2_{L_2(0,T)} \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty |h_{m_1,\dots,m_n}|^2 = C \|\nu\|^2_{L_2(0,T)} \|h\|^2_{L_2(\Omega)},
\end{aligned}$$

where $C>0$ is a constant.

Accordingly, we have

$$\|\nabla u(\cdot,t)\|^2_{L_2(\Omega)} \le C \|\nu\|^2_{L_2(0,T)} \|h\|^2_{L_2(\Omega)}.$$

The function $u(x,t)$ is a generalized solution in the sense of the integral identity (3.5) of monograph [31] follows from Parseval's equality. □

By substituting the expression (2.1) into the integral condition (1.10), we arrive at the following relation:

$$\begin{aligned}
\phi(t) &= \int_\Omega \rho(x)\, u(x,t)\, dx \\
&= \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty \frac{h_{m_1,\dots,m_n}}{1+\lambda_{m_1,\dots,m_n}} \Big( \int_0^t \nu(s)\, e^{-q_{m_1,\dots,m_n}(t-s)}\, ds \Big) \int_\Omega \rho(x)\, w_{m_1,\dots,m_n}(x)\, dx \\
&= \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty \frac{h_{m_1,\dots,m_n}\, \rho_{m_1,\dots,m_n}}{1+\lambda_{m_1,\dots,m_n}} \int_0^t \nu(s)\, e^{-q_{m_1,\dots,m_n}(t-s)}\, ds \\
&= \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty \Psi_{m_1,\dots,m_n} \int_0^t \nu(s)\, e^{-q_{m_1,\dots,m_n}(t-s)}\, ds,
\end{aligned}$$

where

$$\Psi_{m_1,\dots,m_n} = \frac{h_{m_1,\dots,m_n}\,\rho_{m_1,\dots,m_n}}{1+\lambda_{m_1,\dots,m_n}}, \quad m_k \in \mathbb{N}, \quad k=1,\dots,n, \tag{2.2}$$

and $h_{m_1,\dots,m_n}$, $\rho_{m_1,\dots,m_n}$ are the Fourier coefficients corresponding to the functions $h(x)$ and $\rho(x)$, respectively.

We now introduce the function

$$K(t) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \Psi_{m_1,\dots,m_n}\, e^{-q_{m_1,\dots,m_n} t}, \quad t>0. \tag{2.3}$$

As a result, the control problem reduces to the following Volterra-type integral equation of the first kind:

$$\int_0^t K(t-s)\,\nu(s)\,ds = \phi(t), \quad t>0. \tag{2.4}$$

## 3. Main estimates

In this section, we derive the necessary analytical estimates to study the behavior of the kernel function appearing in the Volterra integral equation obtained in the previous section. These estimates play a crucial role in proving the existence and admissibility of the control function, as well as ensuring the well-posedness of the corresponding integral equation.

**Lemma 3.1** (see [32]). *Let $\psi(x_k)$ be a function defined on $[0,\infty)$ such that*

$$\psi(x_k) \ge 0 \quad \text{and} \quad \frac{d\psi}{dx_k} \le 0 \quad \text{for all } x_k \in [0,\infty).$$

*Then, for each $m_k \in \mathbb{N}$, $k=1,\dots,n$, the following inequality holds:*

$$\int_0^{m_k\pi} \psi(x_k)\,\sin x_k\,dx_k \ge 0.$$

**Lemma 3.2.** *Suppose that a function $P(x_1,\dots,x_n)$ is defined on the domain $[0,\infty)^n$ and satisfies the following conditions:*

$$P(x_1,\dots,x_n) \ge 0, \quad \frac{\partial P}{\partial x_k} \le 0 \quad \text{for all } k=1,\dots,n,$$

*then the following inequality is valid:*

$$\int_0^{m_1\pi} \cdots \int_0^{m_n\pi} P(x_1,\dots,x_n)\,\sin x_1 \cdots \sin x_n\,dx_1 \cdots dx_n \ge 0.$$

P r o o f. We apply mathematical induction on the number of spatial variables $n$.

**Step 1:** For $n=1$, the inequality

$$\int_0^{m_1\pi} P(x_1)\sin x_1\,dx_1 \ge 0$$

follows immediately from Lemma 4.1.

**Step 2:** Suppose the Lemma holds for dimension $n-1$. Consider the $n$-dimensional case. Define

$$I_m := \int_0^{m_n\pi} \Big( \int_0^{m_1\pi} \cdots \int_0^{m_{n-1}\pi} P(x_1,\dots,x_n)\sin x_1 \cdots \sin x_{n-1}\,dx_1 \cdots dx_{n-1} \Big) \sin x_n\,dx_n.$$

Let us denote the inner integral by

$$E(x_n) \coloneqq \int_0^{m_1\pi} \cdots \int_0^{m_{n-1}\pi} P(x_1, \dots, x_n) \sin x_1 \cdots \sin x_{n-1}\, dx_1 \cdots dx_{n-1}.$$

By the induction hypothesis and the assumptions on $P$, it holds that

$$E(x_n) \geq 0 \quad \text{and} \quad \frac{dE}{dx_n} \leq 0.$$

Indeed,

$$\frac{dE}{dx_n} = \int_0^{m_1\pi} \cdots \int_0^{m_{n-1}\pi} \frac{\partial P}{\partial x_n}(x_1, \dots, x_n) \sin x_1 \cdots \sin x_{n-1}\, dx_1 \cdots dx_{n-1} \leq 0,$$

since $P$ is decreasing in each $x_k$ by assumption.

Now, applying Lemma 4.1 to $E(x_n)$, we conclude that

$$I_m = \int_0^{m_n\pi} E(x_n) \sin x_n \, dx_n \geq 0.$$

This completes the induction step and proves the Lemma. □

**Corollary 3.1.** *Let $\rho \in \mathcal{A}(\Omega)$. Then the Fourier coefficients of $\rho$ with respect to the orthonormal system $\{w_{m_1,\dots,m_n}(x)\}$ satisfy the non-negativity condition*

$$\rho_{m_1,\dots,m_n} \geq 0, \quad \textit{for all} \quad m_k \in \mathbb{N}, \quad k = \overline{1, n}.$$

P r o o f. The conclusion follows immediately from Lemma 3.2, by taking $P(x_1, \dots, x_n) = \rho(x_1, \dots, x_n)$. Since $\rho \in \mathcal{A}(\Omega)$, the function $\rho$ is non-negative and non-increasing in each variable $x_k$ over $[0, p_k]$. Therefore, the conditions of Lemma 3.2 are satisfied, which implies the desired non-negativity of the Fourier coefficients $\rho_{m_1,\dots,m_n}$. □

**Lemma 3.3.** *Let $\rho \in \mathcal{A}(\Omega)$. Then the following estimate for the Fourier coefficients of $\rho$ holds:*

$$|\rho_{m_1,\dots,m_n}| \leq C\, \lambda_{m_1,\dots,m_n}^{-1/2} \|\nabla \rho\|_{L_2(\Omega)}, \quad m_k \in \mathbb{N}, \quad k = 1, \dots, n,$$

*where $\rho_{m_1,\dots,m_n} = (\rho, w_{m_1,\dots,m_n})$ denotes the inner product in $L_2(\Omega)$, $\lambda_{m_1,\dots,m_n}$ are the eigenvalues corresponding to the eigenfunctions $w_{m_1,\dots,m_n}$ of the operator $\mathcal{L}_n$, and $C > 0$ is a constant.*

P r o o f. By (1.6), we may write

$$\begin{aligned}
\lambda_{m_1,\dots,m_n} \rho_{m_1,\dots,m_n} &= \lambda_{m_1,\dots,m_n} \int_\Omega \rho(x)\, w_{m_1,\dots,m_n}(x)\, dx = -\int_\Omega \rho(x)\, \mathcal{L}_n\, w_{m_1,\dots,m_n}(x)\, dx \\
&= -\sum_{j=0}^{2^n-1} a_j \int_\Omega \rho(x)\, \Delta w_{m_1,\dots,m_n}(S_1^{j_1} \dots S_n^{j_n} x)\, dx \\
&= \sum_{j=0}^{2^n-1} a_j \sum_{i=1}^{n} (-1)^{j_i} \int_\Omega \frac{\partial}{\partial x_i} w_{m_1,\dots,m_n}\left(S_1^{j_1} \dots S_i^{j_i} \dots S_n^{j_n} x\right) \frac{\partial}{\partial x_i} \rho(x)\, dx.
\end{aligned}$$

Note that

$$w_{m_1,\dots,m_n}(S_1^{j_1} \dots S_n^{j_n} x) = C_n \prod_{k=1}^{n} \sin \frac{m_k \pi}{p_k} S_k^{j_k} x_k = C_n \prod_{k=1}^{n} (-1)^{(m_k+1)j_k} \sin \frac{m_k \pi}{p_k} x_k.$$

Then we have

$$\frac{\partial}{\partial x_i} w_{m_1,\dots,m_n}\left(S_1^{j_1}\dots S_n^{j_n}x\right) = C_n\,(-1)^{(m_i+1)j_i}\,\frac{m_i\,\pi}{p_i}\,\cos\frac{m_i\,\pi}{p_i}x_i \prod_{k=1,k\neq i}^{n}(-1)^{(m_k+1)j_k}\sin\frac{m_k\,\pi}{p_k}x_k$$

$$= \Big(\prod_{k=1}^{n}(-1)^{(m_k+1)j_k}\Big)\,\frac{\partial}{\partial x_i} w_{m_1,\dots,m_n}(x) = C_{i,j}\,\frac{\partial}{\partial x_i} w_{m_1,\dots,m_n}(x).$$

It is known that

$$\|\nabla w_{m_1,\dots,m_n}\|^2_{L_2(\Omega)} = (\nabla w_{m_1,\dots,m_n}, \nabla w_{m_1,\dots,m_n}) = (w_{m_1,\dots,m_n}, -\Delta w_{m_1,\dots,m_n}) = \mu_{m_1,\dots,m_n},$$

where $\mu_{m_1,\dots,m_n}$ is defined by (1.8).

Therefore,

$$|\lambda_{m_1,\dots,m_n}\,\rho_{m_1,\dots,m_n}| \le \Big(\sum_{j=0}^{2^n-1}|a_j|\Big)\,\|\nabla\rho\|_{L_2(\Omega)}\,\|\nabla w_{m_1,\dots,m_n}\|_{L_2(\Omega)}$$

$$= \Big(\sum_{j=0}^{2^n-1}|a_j|\Big)\,\sqrt{\mu_{m_1,\dots,m_n}}\,\|\nabla\rho\|_{L_2(\Omega)}.$$

Using $\lambda_{m_1,\dots,m_n} = \theta_{m_1,\dots,m_n}\,\mu_{m_1,\dots,m_n} > 0$, we get the required estimate

$$|\rho_{m_1,\dots,m_n}| \le C\lambda^{-1/2}_{m_1,\dots,m_n}\,\|\nabla\rho\|_{L_2(\Omega)}.$$

□

Now we give the Lemma regarding the continuity of the kernel of the main Volterra integral equation.

**Lemma 3.4.** *Suppose that* $\theta_{m_1,\dots,m_n} > 0$ *for all indices* $m_k \in \mathbb{N}$, $k = \overline{1,n}$. *Let* $h \in L_2(\Omega)$ *and* $\rho \in \mathcal{A}(\Omega)$. *Then the kernel function* $K(t)$, *defined by* (2.3), *is continuous for all* $t \ge 0$.

P r o o f. From Lemma 3.2 and the condition (1.9), we know that the product $\Psi_{m_1,\dots,m_n} \ge 0$ for all indices. Hence, the kernel

$$K(t) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\Psi_{m_1,\dots,m_n}\,e^{-q_{m_1,\dots,m_n}t}$$

is strictly positive and monotonically decreasing for $t > 0$, i. e., $K(t) > 0$, $K'(t) < 0$, for all $t > 0$.

Now, using Lemma 3.3 and the definition of $\Psi_{m_1,\dots,m_n}$ in (2.2), we estimate:

$$\Psi_{m_1,\dots,m_n} = \frac{h_{m_1,\dots,m_n}\,\rho_{m_1,\dots,m_n}}{1+\lambda_{m_1,\dots,m_n}} \le C\,\frac{h_{m_1,\dots,m_n}}{1+\lambda_{m_1,\dots,m_n}}\cdot\lambda^{-1/2}_{m_1,\dots,m_n}\,\|\nabla\rho\|_{L_2(\Omega)},$$

where $C > 0$ is a constant independent of $m_1,\dots,m_n$.

Substituting this estimate into the series expression for $K(t)$, we obtain

$$K(t) \le C\,\|\nabla\rho\|_{L_2(\Omega)}\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\frac{h_{m_1,\dots,m_n}}{(1+\lambda_{m_1,\dots,m_n})\sqrt{\lambda_{m_1,\dots,m_n}}}\,e^{-q_{m_1,\dots,m_n}t}$$

$$\le C\,\|\nabla\rho\|_{L_2(\Omega)}\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\frac{h_{m_1,\dots,m_n}}{(1+\lambda_{m_1,\dots,m_n})\sqrt{\lambda_{m_1,\dots,m_n}}},$$

which converges due to the square integrability of $h(x)$ and the growth of eigenvalues $\lambda_{m_1,\dots,m_n} \to \infty$. □

## 4. Proof of Theorem 1.1

In this section, we establish the existence of a solution to the Volterra integral equation of the first kind by employing the Laplace transform method. Furthermore, we demonstrate that the corresponding control function satisfies the admissibility condition.

The Laplace transform of the function $\nu(t)$ is defined by

$$\widetilde{\nu}(p) = \int_0^\infty e^{-pt}\nu(t)\,dt,$$

where $p = \delta + i\zeta$ with $\delta > 0$ and $\zeta \in \mathbb{R}$.

Applying the Laplace transform to both sides of the Volterra equation (2.4), and using the convolution theorem, we obtain the relation

$$\widetilde{\phi}(p) = \int_0^\infty e^{-pt}\Big(\int_0^t K(t-s)\nu(s)\,ds\Big)dt = \widetilde{K}(p)\cdot\widetilde{\nu}(p). \tag{4.1}$$

Consequently, we obtain

$$\widetilde{\nu}(p) = \frac{\widetilde{\phi}(p)}{\widetilde{K}(p)},$$

and the inverse transform yields

$$\nu(t) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\widetilde{\phi}(p)}{\widetilde{K}(p)}e^{pt}\,dp = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{\phi}(\delta+i\zeta)}{\widetilde{K}(\delta+i\zeta)}e^{(\delta+i\zeta)t}\,d\zeta. \tag{4.2}$$

**Lemma 4.1.** *The following estimate is valid:*

$$|\widetilde{K}(\delta+i\,\zeta)| \geq \frac{C_\delta}{\sqrt{1+\zeta^2}}, \qquad \delta > 0, \qquad \zeta\in\mathbb{R},$$

*where $C_\delta > 0$ is a constant only depending on $\delta$.*

P r o o f. The Laplace transform of the kernel $K(t)$ is expressed as

$$\widetilde{K}(p) = \int_0^\infty K(t)e^{-pt}\,dt = \sum_{m_1=1}^\infty\cdots\sum_{m_n=1}^\infty \Psi_{m_1,\dots,m_n}\int_0^\infty e^{-(p+q_{m_1,\dots,m_n})t}\,dt$$
$$= \sum_{m_1=1}^\infty\cdots\sum_{m_n=1}^\infty \frac{\Psi_{m_1,\dots,m_n}}{p+q_{m_1,\dots,m_n}}.$$

For $p = \delta + i\zeta$, this becomes

$$\widetilde{K}(\delta+i\zeta) = \sum_{m_1=1}^\infty\cdots\sum_{m_n=1}^\infty \frac{\Psi_{m_1,\dots,m_n}}{\delta+q_{m_1,\dots,m_n}+i\zeta} = \sum_{m_1=1}^\infty\cdots\sum_{m_n=1}^\infty \frac{\Psi_{m_1,\dots,m_n}(\delta+q_{m_1,\dots,m_n})}{(\delta+q_{m_1,\dots,m_n})^2+\zeta^2}$$
$$- i\zeta\sum_{m_1=1}^\infty\cdots\sum_{m_n=1}^\infty \frac{\Psi_{m_1,\dots,m_n}}{(\delta+q_{m_1,\dots,m_n})^2+\zeta^2} = \operatorname{Re}\widetilde{K}(\delta+i\zeta) + i\ \operatorname{Im}\widetilde{K}(\delta+i\zeta),$$

where

$$\operatorname{Re}\widetilde{K}(\delta+i\zeta) = \sum_{m_1=1}^\infty\cdots\sum_{m_n=1}^\infty \frac{\Psi_{m_1,\dots,m_n}(\delta+q_{m_1,\dots,m_n})}{(\delta+q_{m_1,\dots,m_n})^2+\zeta^2},$$

and

$$\operatorname{Im} \widetilde{K}(\delta + i\zeta) = -\zeta \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{\Psi_{m_1,\dots,m_n}}{(\delta + q_{m_1,\dots,m_n})^2 + \zeta^2}.$$

Observing the inequality

$$(\delta + q_{m_1,\dots,m_n})^2 + \zeta^2 \le \left((\delta + q_{m_1,\dots,m_n})^2 + 1\right)(1 + \zeta^2),$$

we deduce

$$\frac{1}{(\delta + q_{m_1,\dots,m_n})^2 + \zeta^2} \ge \frac{1}{1+\zeta^2} \cdot \frac{1}{(\delta + q_{m_1,\dots,m_n})^2 + 1}.$$

This leads to estimates for the real and imaginary parts of $\widetilde{K}(\delta + i\zeta)$:

$$\begin{aligned} |\operatorname{Re} \widetilde{K}(\delta + i\zeta)| &= \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{\Psi_{m_1,\dots,m_n}(\delta + q_{m_1,\dots,m_n})}{(\delta + q_{m_1,\dots,m_n})^2 + \zeta^2} \\ &\ge \frac{1}{1+\zeta^2} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{\Psi_{m_1,\dots,m_n}(\delta + q_{m_1,\dots,m_n})}{(\delta + q_{m_1,\dots,m_n})^2 + 1} = \frac{C_{1,\delta}}{1+\zeta^2}, \end{aligned}$$

and

$$\begin{aligned} |\operatorname{Im} \widetilde{K}(\delta + i\zeta)| &= |\zeta| \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{\Psi_{m_1,\dots,m_n}}{(\delta + q_{m_1,\dots,m_n})^2 + \zeta^2} \\ &\ge \frac{|\zeta|}{1+\zeta^2} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{\Psi_{m_1,\dots,m_n}}{(\delta + q_{m_1,\dots,m_n})^2 + 1} = \frac{C_{2,\delta}|\zeta|}{1+\zeta^2}, \end{aligned}$$

where

$$C_{1,\delta} = \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{\Psi_{m_1,\dots,m_n}(\delta + q_{m_1,\dots,m_n})}{(\delta + q_{m_1,\dots,m_n})^2 + 1}, \quad C_{2,\delta} = \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{\Psi_{m_1,\dots,m_n}}{(\delta + q_{m_1,\dots,m_n})^2 + 1}.$$

Combining these results, we obtain

$$|\widetilde{K}(\delta + i\zeta)|^2 = |\operatorname{Re} \widetilde{K}(\delta + i\zeta)|^2 + |\operatorname{Im} \widetilde{K}(\delta + i\zeta)|^2 \ge \frac{\min(C_{1,\delta}^2, C_{2,\delta}^2)}{1+\zeta^2},$$

yielding

$$|\widetilde{K}(\delta + i\zeta)| \ge \frac{C_\delta}{\sqrt{1+\zeta^2}}, \tag{4.3}$$

where $C_\delta = \min(C_{1,\delta}, C_{2,\delta})$. □

Taking the limit as $\delta \to 0$, (4.2) simplifies to

$$\nu(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\widetilde{\phi}(i\zeta)}{\widetilde{K}(i\zeta)} e^{i\zeta t}\, d\zeta. \tag{4.4}$$

To establish admissibility, the following lemma is required.

**Lemma 4.2** (see [33])**.** *Suppose* $\phi(t) \in W(M)$*. Then, for the imaginary part of its Laplace transform, the following estimate holds*

$$\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|\sqrt{1+\zeta^2}\, d\zeta \le C_1 \|\phi\|_{W_2^2(\mathbb{R}_+)}$$

*where* $C_1 > 0$ *is a constant.*

**Lemma 4.3.** *Assume that* $\phi \in W_2^1(-\infty, +\infty)$ *and* $\phi(t) = 0$ *for* $t \le 0$*. Then* $\nu \in L_2(\mathbb{R}_+)$*.*

P r o o f. Using (4.1) and (4.3), we have

$$\begin{aligned}\|\nu\|_{L_2(\mathbb{R})}^2 &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\widetilde{\nu}(\zeta)|^2\, d\zeta = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\frac{\widetilde{\phi}(i\zeta)}{\widetilde{K}(i\zeta)}\right|^2 d\zeta \\ &\le \frac{C_1}{2\pi}\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|^2 (1+|\zeta|^2)\, d\zeta = \ C\,\|\phi\|_{W_2^1(\mathbb{R})}^2,\end{aligned}$$

where $C = C_1/(2\pi)$. □

P r o o f of **Theorem 1.1.** From (4.3), (4.4) and Lemma 4.2, we have the estimate

$$\begin{aligned}|\nu(t)| &\le \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|\widetilde{\phi}(i\zeta)|}{|\widetilde{K}(i\zeta)|} d\zeta \le \frac{1}{2\pi C_0}\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|\sqrt{1+\zeta^2}\, d\zeta \\ &\le \frac{C_1}{2\pi C_0}\|\phi\|_{W_2^2(R_+)} \le \frac{C_1\, M}{2\pi C_0} = 1,\end{aligned}$$

where $C_0 = \min(C_{1,0}, C_{2,0})$ which is defined by (4.3), and $M = 2\pi C_0/C_1$.

Theorem 1.1 is proved.

## References


[1] B. D. Coleman, W. Noll, "An approximation theorem for functionals, with applications in continuum mechanics", *Arch. Rational Mech. Anal.*, **6** (1960), 355–370.

[2] P. Chen, M. Gurtin, "On a theory of heat conduction involving two temperatures", *Z. Angew. Math. Phys.*, **19** (1968), 614–627.

[3] L. W. White, "Controllability properties of pseudo-parabolic boundary control problems", *SIAM J. Control and Optimization*, **18** (1980), 534–539.

[4] B. D. Coleman, R. J. Duffin, V. J. Mizel, "Instability, uniqueness, and nonexistence theorems for the equation on a strip", *Arch. Rational Mech. Anal.*, **19** (1965), 100–116.

[5] L. W. White, "Point control of pseudoparabolic problems", *Journal of Differential Equations*, **42**:3 (1981), 366–374.

[6] A. Friedman, "Optimal control for parabolic equations", *J. Math. Anal. Appl.*, **18**:3 (1967), 479–491.

[7] H. O. Fattorini, D. L. Russell, "Exact controllability theorems for linear parabolic equations in one space dimension", *Arch. Rational Mech. Anal.*, **43** (1971), 272–292.

[8] Yu. V. Egorov, "The optimal control in Banach space", *Uspekhi Mat. Nauk.*, **18** (1963), 211–213.

[9] A. V. Fursikov, *Optimal Control of Distributed Systems. Theory and Applications*: *Translations of Mathematical Monographs*, **187**, Amer. Math. Soc., Providence, Rhode Island, 2000, 305 pp.

[10] J. L. Lions, *Contróle optimal de systèmes gouvernés par des équations aux dérivées partielles*, Dunod Gauthier–Villars, Paris, 1968.

[11] G. Schmidt, “The “Bang–Bang” principle for the time-optimal problem in boundary control of the heat equation”, *SIAM Journal on Control and Optimization*, **18** (1980), 101–107.

[12] S. Albeverio, Sh. A. Alimov, “On one time-optimal control problem associated with the heat exchange process”, *Appl. Math. Opt.*, **57** (2008), 58–68.

[13] Z. K. Fayazova, “Boundary control of the heat transfer process in the space”, *Russian Mathematics*, **12** (2019), 82–90.

[14] F. N. Dekhkonov, “On the time-optimal control problem for a heat equation”, *Bulletin of the Karaganda University Mathematics Series*, **111** (2023), 28–38.

[15] B. Allal, G. Fragnelli, J. Salhi, “Null controllability for degenerate parabolic equations with a nonlocal space term”, *Discrete and Continuous Dynamical Systems - S*, **17** (2024), 1821–1856.

[16] F. Dekhkonov, “On one boundary control problem for a pseudo-parabolic equation in a two-dimensional domain”, *Communications in Analysis and Mechanics*, **17** (2025), 1–14.

[17] F. N. Dekhkonov, “On the control problem associated with a pseudo-parabolic type equation in an one-dimensional domain”, *International Journal of Applied Mathematics*, **37**:1 (2024), 109–118.

[18] B. K. Turmetov, B. J. Kadirkulov, “An inverse problem for a parabolic equation with involution”, *Lobachevskii J. of Math.*, **42** (2021), 3006–3015.

[19] B. K. Turmetov, B. J. Kadirkulov, “On the solvability of an initial-boundary value problem for a fractional heat equation with involution”, *Lobachevskii J. Math.*, **43** (2022), 249–262.

[20] E. Mussirepova, A. Sarsenbi, A. Sarsenbi, “The inverse problem for the heat equation with reflection of the argument and with a complex coefficient”, *Bound Value Probl.*, **1** (2022), 99.

[21] B. Ahmad, A. Alsaedi, M. Kirane, R. Tapdigoglu, “An inverse problem for space and time fractional evolution equations with an involution perturbation”, *Quaest. Math.*, **40** (2017), 151–160.

[22] A. Kopzhassarova, A. Sarsenbi, “Basis properties of eigenfunctions of second-order differential operators with involution”, *Abstr. Appl. Anal.*, 2012, 576843.

[23] M. Kirane, A. A. Sarsenbi, “Solvability of mixed problems for a fourth-order equation with involution and fractional derivative”, *Fractal Fract.*, **7** (2023), 131.

[24] B. Turmetov, V. Karachik, “On solvability of some inverse problems for a nonlocal fourth-order parabolic equation with multiple involution”, *AIMS Mathematics*, **9** (2024), 6832–6849.

[25] M. Muratbekova, B. Kadirkulov, M. Koshanova, B. Turmetov, “On solvability of some inverse problems for a fractional parabolic equation with a nonlocal biharmonic operator”, *Fractal and Fractional*, **7**:5 (2023), 404.

[26] N. Al-Salti, M. Kirane, B. T. Torebek, “On a class of inverse problems for a heat equation with involution perturbation”, *Hacet. J. Math. Stat.*, **48** (2019), 669–681.

[27] F. N. Dekhkonov, “On the control problem for a heat conduction equation with involution in a two-dimensional domain”, *Lobachevskii J. Math.*, **46**:2 (2025), 613–623.

[28] F. N. Dekhkonov, “Boundary control problem for a parabolic equation with involution”, *Eurasian Journal of Mathematical and Computer Applications*, **12** (2024), 22–34.

[29] S. G. Mikhlin, *Linear Partial Differential Equations*, Vysshaya Shkola Publ., Moscow, 1977.

[30] B. Turmetov, V. Karachik, “On eigenfunctions and eigenvalues of a nonlocal Laplace operator with involution in a parallelepiped”, *AIP Conf. Proc.*, Sixth International Conference of Mathematical Sciences (ICMS 2022) (Istanbul, Turkey, 20–24 July 2022), **2879**, 2023.

[31] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva, *Linear and Quasi-Linear Equations of Parabolic Type*, Nauka Publ., Moscow, 1967.

[32] Sh. A. Alimov, F. N. Dekhkonov, “On a control problem associated with fast heating of a thin rod”, *Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences*, **2** (2019), 1–14.

[33] F. Dekhkonov, W. Li, “On the boundary control problem associated with a fourth order parabolic equation in a two-dimensional domain”, *Discrete and Continuous Dynamical Systems - S*, **17** (2024), 2478–2488.

**Information about the authors**

**Farrukh N. Dekhkonov**, Candidate of Physics and Mathematics, Associate Professor of the Mathematical Analysis Department. Namangan State University, Namangan, Uzbekistan. E-mail: f.n.dehqonov@mail.ru
**ORCID:** https://orcid.org/0000-0003-4747-8557

**Batirkhan Kh. Turmetov**, Doctor of Physics and Mathematics, Professor, Department of Mathematics, Khoja Akhmet Yassawi International Kazakh-Turkish University, Turkistan, Kazakhstan; Professor, Department of Mathematics and Physics, Alfraganus University, Tashkent, Uzbekistan.
E-mail: batirkhan.turmetov@ayu.edu.kz
**ORCID:** https://orcid.org/0000-0001-7735-6484

**Информация об авторах**

**Дехконов Фаррух Нуриддин угли**, кандидат физико-математических наук, доцент кафедры математического анализа. Наманганский государственный университет, г. Наманган, Узбекистан. E-mail: f.n.dehqonov@mail.ru
**ORCID:** https://orcid.org/0000-0003-4747-8557

**Турметов Батирхан Худайбергенович**, доктор физико-математических наук, профессор кафедры математики, Международный казахско-турецкий университет имени Ходжи Ахмеда Ясави, г. Туркестан, Казахстан; профессор кафедры математики и физики, Университет Альфраганус, г. Ташкент, Узбекистан.
E-mail: batirkhan.turmetov@ayu.edu.kz
**ORCID:** https://orcid.org/0000-0001-7735-6484

There is no conflict of interests.

Конфликт интересов отсутствует.

**Corresponding author:**
Batirkhan Kh. Turmetov
E-mail: batirkhan.turmetov@ayu.edu.kz

**Для контактов:**
Турметов Батирхан Худайбергенович
E-mail: batirkhan.turmetov@ayu.edu.kz

Received 02.05.2025
Reviewed 14.09.2025
Accepted for press 21.11.2025

Поступила в редакцию 02.05.2025 г.
Поступила после рецензирования 14.09.2025 г.
Принята к публикации 21.11.2025 г.