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Pólya groups and fields in some real biquadratic number fields

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Abstract. Let K be a number field and $\mathcal{O}_K$ be its ring of integers. Let $\prod_q(K)$ be the product of all prime ideals of $\mathcal{O}_K$ with absolute norm q. The Pólya group of a number field K is the subgroup of the class group of K generated by the classes of $\prod_q(K)$. K is a Pólya field if and only if the ideals $\prod_q(K)$ are principal. In this paper, we follow the work that we have done in [S. EL Madrari, "On the Pólya fields of some real biquadratic fields", Matematicki Vesnik, online 05.09.2024] where we studied the Pólya groups and fields in a particulare cases. Here, we will give the Pólya groups of $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that $d_1 = lm_1$ and $d_2 = lm_2$ are square-free integers with $l > 1$ and $gcd(m_1, m_2) = 1$ and the prime 2 is not totally ramified in $K/\mathbb{Q}$. And then, we characterize the Pólya fields of the real biquadratic fields K.

Keywords: Pólya fields, Pólya groups, real biquadratic fields, the first cohomology group of units, integer-valued polynomials

Mathematics Subject Classification: 11R04, 11R16, 11R27, 13F20.

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НАУЧНАЯ СТАТЬЯ

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Группы и поля Пойи в некоторых действительных биквадратичных числовых полях

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Аннотация. Пусть K — числовое поле, а $\mathcal{O}_K$ — его кольцо целых чисел. Пусть $\prod_q(K)$ — произведение всех простых идеалов $\mathcal{O}_K$ с абсолютной нормой q. Группа Пойи числового поля K — это подгруппа группы классов K, порожденная классами $\prod_q(K)$. K является полем Пойи тогда и только тогда, когда идеалы $\prod_q(K)$ являются главными. В этой статье мы следуем нашей работе [S. EL Madrari, "On the Pólya fields of some real biquadratic fields" Matematicki Vesnik, online 05.09.2024], в которой мы изучали группы и поля Пойи в частных случаях. Здесь мы дадим группы Пойи $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ такие, что $d_1 = lm_1$ и $d_2 = lm_2$ являются свободными от квадратов целыми числами с $l > 1$ и НОД$(m_1, m_2) = 1$, а простое число 2 не полностью разветвлено в $K/\mathbb{Q}$. А затем мы охарактеризуем поля Пойи действительных биквадратичных полей K.

Ключевые слова: поля Пойи, группы Пойи, действительные биквадратичные поля, первая когомологическая группа единиц, целочисленные многочлены

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Introduction

Let K be a number field and $\mathcal{O}_K$ be its ring of integers. Let $\operatorname{Int}(\mathcal{O}_K) = \{R \in K[X] \mid R(\mathcal{O}_K) \subset \mathcal{O}_K\}$ be the ring of integer-valued polynomials on $\mathcal{O}_K$. According to Pólya in [1], a basis $(g_n)_{n\in\mathbb{N}}$ of $\operatorname{Int}(\mathcal{O}_K)$ is said to be a regular basis if the deg $(g_n) = n$ for each polynomial g_n. In 1919, G. Pólya was interested whether the $\mathcal{O}_K$-module $\operatorname{Int}(\mathcal{O}_K)$ has a regular basis. Ostrowski [2] showed that the $\mathcal{O}_K$-module $\operatorname{Int}(\mathcal{O}_K)$ admits a regular basis if and only if the ideals $\prod_q(K)$ are principal, where $\prod_q(K)$ is the product of all prime ideals of $\mathcal{O}_K$ with absolute norm q. In 1982, Zantema in [3] gave the name of Pólya field to any field K such that the $\mathcal{O}_K$-module $\operatorname{Int}(\mathcal{O}_K)$ has a regular basis. In 1997, Cahen and Chabert in [4] introduced the notion of Pólya group which is the group generated by the classes of $\prod_q(K)$.

Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that $d_1 = lm_1$ and $d_2 = lm_2$ are square-free integers with $l > 1$ and $gcd(m_1, m_2) = 1$. The studies about the Pólya fields in the real biquadratic fields started in 1982 by Zantema [3]. In 2011, A. Leriche in [5] gave some Pólya fields of K by using the capitulation. Otheres (see [6], [7], and [8]) determined some particular cases of Pólya groups and Pólya fields of K.

In this paper, we are going to determine $H^1(G_K, E_K)$ which is the first cohomology group of units of $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that $d_1 = lm_1$ and $d_2 = lm_2$ are square-free integers with $l > 1$ and $gcd(m_1, m_2) = 1$ and the prime 2 is not totally ramified in $K/\mathbb{Q}$. And then, we give the Pólya groups of K. Lastly, we give the Pólya fields of the real biquadratic fields K. This paper continues the study of [9].

1. Notations

In this work, we adopt the following notations:

- $l > 1$ and $m_1 > 1$ and $m_2 > 1$ are square-free integers.
- $d_1 = lm_1$ and $d_2 = lm_2$ and $d_3 = m_1 m_2$ are square-free integers.
- $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$: a real biquadratic number field.
- $\mathcal{O}_K$: the ring of integers of K.
- $k_i = \mathbb{Q}(\sqrt{d_i})$: the quadratic subfields of K for $i = 1, 2, 3$.
- $\epsilon_i = x_i + y_i\sqrt{d_i}$: the fundamental unit of $\mathbb{Q}(\sqrt{d_i})$, for $i = 1, 2, 3$.
- $N(\eta_i) = N_i(\eta_i) = Norm_{k_i/\mathbb{Q}}(\eta_i)$ where $\eta_i \in k_i$, for $i = 1, 2, 3$.
- E_K : the unit group of K over $\mathbb{Q}$.
- G_K : the Galois group of K over $\mathbb{Q}$.
- e_p : the ramification index of a prime number p in $K/\mathbb{Q}$.
- d_K : the discriminant of K over $\mathbb{Q}$.
- t : the number of the prime divisors of d_K.

2. Preliminaries

D e f i n i t i o n 2.1. Let $\prod_q(L)$ be the product of all prime ideals of $\mathcal{O}_L$ with norm $q \geq 2$. The Pólya group $\mathcal{P}_O(L)$ of a number field L is the subgroup of the class group of L generated by the classes of the ideals $\prod_q(L)$.

In the real biquadratic number fields K, the prime 2 is the only prime can be totally ramified in $K/\mathbb{Q}$. When e_2 the ramification index of the prime 2 in $K/\mathbb{Q}$ is $4 = [K : \mathbb{Q}]$, in other words 2 is totally ramified in $K/\mathbb{Q}$ so we have $(d_1, d_2) \equiv (2, 3)$ or $(3, 2) \pmod 4$, therefore $N\epsilon_1 \neq N\epsilon_2 = N\epsilon_3 = 1$, $N\epsilon_2 \neq N\epsilon_1 = N\epsilon_3 = 1$, or $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$. When $e_2 \neq 4$, i. e., the prime 2 is not totally ramified in $K/\mathbb{Q}$. So, we have either $e_2 = 1$, when the prime 2 is not ramified in $K/\mathbb{Q}$ or $e_2 = 2$, when the prime 2 is ramified in $K/\mathbb{Q}$. Thus, we have the following possibilities $(d_1, d_2) \equiv (1,1),(1,2),(2,1),(1,3),(3,1),(3,3) \pmod 4$. Let $k_j = \mathbb{Q}(\sqrt{d_j})$, $j = 1,2$ note that when $d_j \equiv 1,2 \pmod 4$, then $N\epsilon_j = \pm 1$, for $j = 1,2$ and when there exists a prime number $\equiv 3 \pmod 4$ dividing d_j then $N\epsilon_j = +1$, for $j = 1,2$.

Let $K = \mathbb{Q}(\sqrt{d}_1, \sqrt{d}_2)$ such that $d_1 = lm_1$ and $d_2 = lm_2$ are two square-free integers with $l > 1$ and $gcd(m_1, m_2) = 1$. Let $H^1(G_K, E_K)$ be the first cohomology group of units of K. Let $\epsilon_i = x_i + y_i\sqrt{d_i}$ be the fundamental unit of $\mathbb{Q}(\sqrt{d_i})$, for $i = 1,2,3$. Recall that $a_i \in \mathbb{Q}$ such that $a_i = N(\epsilon_i + 1) = 2(x_i + 1)$ when $N\epsilon_i = 1$ else $a_i = 1$, for $i = 1,2,3$. Let H be the subgroup of $\mathbb{Q}^*/\mathbb{Q}^{*2}$ generated by the images of d_1, d_2, d_3, a_1, a_2 and a_3 with $d_1 = lm_1$, $d_2 = lm_2$, and $d_3 = m_1m_2$. $[a_i]$ is the class of a_i in $\mathbb{Q}^*/\mathbb{Q}^{*2}$, for $i = 1,2,3$ and $[d_i]$ is the class of d_i in $\mathbb{Q}^*/\mathbb{Q}^{*2}$, for $i = 1,2,3$.

Theorem 2.1 (see [10]). *$H \simeq H^1(G_K, E_K)$, except for the next two cases in which H is canonically isomorphic to a subgroup of index 2 of $H^1(G_K, E_K)$:*

1. *the prime 2 is totally ramified in $K/\mathbb{Q}$, and there exists integral $z_i \in k_i$, $i = 1,2,3$ such that $N_1(z_1) = N_2(z_2) = N_3(z_3) = \pm 2$,*
2. *all the quadratic subfields k_i contain units of norm -1 and $E_K = E_{k_1}E_{k_2}E_{k_3}$.*

R e m a r k 2.1. The theorem above was given by C. Bennett Setzer in [10]. It presents the first cohomology group of units of the real biquadratic number fields K. For the proof of the theorem above, the reader refers to see the proof in [10, Theorems 4,5,7]. Note that the theorem above is mentioned by Zantema in [3, Section 4, p. 14,15], also it is mentioned in [6].

Now we give a well-known proposition in the notion of Pólya group and field (see [5, Proposition 2.3]).

P r o p o s i t i o n 2.1. *The group $\mathcal{P}_O(L)$ is trivial if and only if one of the following assertions is satisfied:*

1. *the field L is a Pólya field,*
2. *all the ideals $\prod_q(L)$ are principal,*
3. *the $\mathcal{O}_L$-module $\mathrm{Int}(\mathcal{O}_L)$ has a regular basis.*

Zantema gave the following proposition which connects the first cohomology group of units of a number field L with the Pólya group of L in a Galois extension.

P r o p o s i t i o n 2.2 (see [3]). *Let $L/\mathbb{Q}$ be a Galois extension and d_L be its discriminant. Denote by e_p the ramification index of a prime number p in L. Then, the following sequence is exact*

$$1 \to H^1(G_L, E_L) \to \oplus_{p|d_L}\mathbb{Z}/e_p\mathbb{Z} \to \mathcal{P}_O(L) \to 1.$$

In particular, $|H^1(G_L, E_L)||\mathcal{P}_O(L)| = \prod_{p|d_L} e_p$.

Hence, we get the following result.

Corollary 2.1. *L is a Pólya field if and only if $|H^1(G_L, E_L)| = \prod_{p|d_L} e_p$.*

Proposition 2.3 (see [11]). *Let* $K=\mathbb{Q}(\sqrt{d}_1,\sqrt{d}_2)$. *Let* ϵ_i *be the fundamental unit of* $\mathbb{Q}(\sqrt{d_i})$, $i=1,2,3$. *Let* E_K *be the unit group of* K *over* $\mathbb{Q}$. *So, we have the following possibilities for a system of fundamental of units of* E_K *:*

1. $\epsilon_i,\epsilon_j,\epsilon_k$,
2. $\sqrt{\epsilon_i},\epsilon_j,\epsilon_k$ *with* $N\epsilon_i=1$,
3. $\sqrt{\epsilon_i},\sqrt{\epsilon_j},\epsilon_k$ *such that* $N\epsilon_i=N\epsilon_j=1$,
4. $\sqrt{\epsilon_i\epsilon_j},\epsilon_j,\epsilon_k$ *such that* $N\epsilon_i=N\epsilon_j=1$,
5. $\sqrt{\epsilon_i\epsilon_j},\sqrt{\epsilon_k},\epsilon_j$ *where* $N\epsilon_i=N\epsilon_j=N\epsilon_k=1$,
6. $\sqrt{\epsilon_i\epsilon_j},\sqrt{\epsilon_j\epsilon_k},\sqrt{\epsilon_k\epsilon_i}$ *where* $N\epsilon_i=N\epsilon_j=N\epsilon_k=1$,
7. $\sqrt{\epsilon_i\epsilon_j\epsilon_k},\epsilon_j,\epsilon_k$ *where* $N\epsilon_i=N\epsilon_j=N\epsilon_k=1$,
8. $\sqrt{\epsilon_i\epsilon_j\epsilon_k},\epsilon_j,\epsilon_k$ *with* $N\epsilon_i=N\epsilon_j=N\epsilon_k=-1$, *where* $\{\epsilon_i,\epsilon_j,\epsilon_k\}=\{\epsilon_3,\epsilon_1,\epsilon_2\}$.

Proposition 2.4 (see [11]). *Let* $k=\mathbb{Q}(\sqrt{d})$ *such that* $N\epsilon=1$ *and let* λ *denote the square-free part of the positive integer* $N(\epsilon+1)$. *Then* $\lambda>1$, λ *divides the discriminant of* k, $\lambda\neq d$, *and* $\sqrt{\lambda\epsilon}\in k$.

3. The Pólya Groups of The Real Biquadratic Fields $K=\mathbb{Q}(\sqrt{lm_1},\sqrt{lm_2})$

In this section, we are going to determine the Pólya groups of the fields K. Firstly, we need to give the first cohomology group of units of K.

3.1. The structure of the first cohomology group of units of $K=\mathbb{Q}(\sqrt{lm}_1,\sqrt{lm}_2)$

Proposition 3.1. *Let* $K=\mathbb{Q}(\sqrt{d}_1,\sqrt{d}_2)$ *such that* $d_1=lm_1$ *and* $d_2=lm_2$ *are square-free integers with* $l>1$ *and* $gcd(m_1,m_2)=1$. *Let* ϵ_1,ϵ_2 *and* ϵ_3 *be the fundamental unit of* $\mathbb{Q}(\sqrt{d_1})$, $\mathbb{Q}(\sqrt{d_2})$ *and* $\mathbb{Q}(\sqrt{d_3})$ *with* $d_3=m_1m_2$ *respectively, and let* $N\epsilon_1=N\epsilon_2=N\epsilon_3=1$. *Let* λ_1,λ_2 *and* λ_3 *be the square-free part of* $N(\epsilon_1+1)$, $N(\epsilon_2+1)$ *and* $N(\epsilon_3+1)$ *respectively. Then, we have the following results:*

1. $\sqrt{\epsilon_1\epsilon_2}\in K$ *if and only if either* $[\lambda_1\lambda_2]=[lm_1],[lm_2]$ *or* $[m_1m_2]$, *or* $\lambda_1=\lambda_2=l$,
2. $\sqrt{\epsilon_j\epsilon_3}\in K$ *for* $j=1$ *or* 2 *if and only if either* $[\lambda_j\lambda_3]=[lm_1],[lm_2]$ *or* $[m_1m_2]$, *or* $\lambda_j=\lambda_3=m_j$,
3. $\sqrt{\epsilon_1\epsilon_2\epsilon_3}\in K$ *if and only if either* $[\lambda_1\lambda_2\lambda_3]=[lm_1],[lm_2]$ *or* $[m_1m_2]$, *or* $[\lambda_1\lambda_2]=[\lambda_3]$.

Proof. Let $k_i=\mathbb{Q}(\sqrt{d_i})$ such that $N\epsilon_i=1$ for $i=1,2,3$ and let λ_i be the square-free part of the positive integer $N(\epsilon_i+1)$ for $i=1,2,3$. Recall that $[lm_1],[lm_2]$ and $[m_1m_2]$ is the class of lm_1,lm_2 and m_1m_2 in $\mathbb{Q}^*/\mathbb{Q}^{*2}$ respectively. We start by the first equivalent.

1. ($\Longrightarrow$), we use the contrapositive. We suppose that $([\lambda_1\lambda_2]\neq[lm_1],[lm_2]$ and $[m_1m_2])$, and $(\lambda_1\neq l$ or $\lambda_2\neq l)$. We know that $\sqrt{\lambda_1\epsilon_1}\in k_1$ and $\sqrt{\lambda_2\epsilon_2}\in k_2$ (see Proposition 2.4), so $\sqrt{\lambda_1\lambda_2\epsilon_1\epsilon_2}\in K$ and since $([\lambda_1\lambda_2]\neq[lm_1],[lm_2]$ and $[m_1m_2])$, and $(\lambda_1\neq l$ or $\lambda_2\neq l)$, so $\sqrt{\epsilon_1\epsilon_2}\notin K$. Reciprocally, we suppose either $[\lambda_1\lambda_2]=[lm_1],[lm_2]$ or $[m_1m_2]$, or $\lambda_1=\lambda_2=l$, and since we have $\sqrt{\lambda_1\epsilon_1}\in k_1$ and $\sqrt{\lambda_2\epsilon_2}\in k_2$. So, $\sqrt{\lambda_1\epsilon_1}\sqrt{\lambda_2\epsilon_2}\in K$ and thus we get that $\sqrt{\epsilon_1\epsilon_2}\in K$.

2. As above the first assertion we get the second.

3. Lastely, ($\Longrightarrow$) assuming that $[\lambda_1\lambda_2\lambda_3]\neq[lm_1],[lm_2]$ and $[m_1m_2]$, and $[\lambda_1\lambda_2]\neq[\lambda_3]$. Since $\sqrt{\lambda_1\epsilon_1}\in k_1$, $\sqrt{\lambda_2\epsilon_2}\in k_2$, and $\sqrt{\lambda_3\epsilon_3}\in k_3$, so $\sqrt{\lambda_1\epsilon_1\lambda_2\epsilon_2\lambda_3\epsilon_3}\in K$. As we have $[\lambda_1\lambda_2\lambda_3]\neq[lm_1],[lm_2]$ and $[m_1m_2]$, and $[\lambda_1\lambda_2]\neq[\lambda_3]$, so $\sqrt{\epsilon_1\epsilon_2\epsilon_3}\notin K$. Now we suppose either $[\lambda_1\lambda_2\lambda_3]=[lm_1],[lm_2]$ or $[m_1m_2]$, or $[\lambda_1\lambda_2]=[\lambda_3]$. As $\sqrt{\lambda_1\epsilon_1}\in k_1$ and $\sqrt{\lambda_2\epsilon_2}\in k_2$ and then $\sqrt{\lambda_3\epsilon_3}\in k_3$, thus we get that $\sqrt{\epsilon_1\epsilon_2\epsilon_3}\in K$. □

E x a m p l e 3.1. In the field $K = \mathbb{Q}(\sqrt{7\cdot 5}, \sqrt{7\cdot 11})$, we have $d_1 = 7\cdot 5 = 35$, $d_2 = 7\cdot 11 = 77$ and $d_3 = 5\cdot 11 = 55$. The fundamental units are $\epsilon_1 = 6+\sqrt{35}$, $\epsilon_2 = \frac{1}{2}(9+\sqrt{77})$, $\epsilon_3 = 89+12\sqrt{55}$ such that $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$. So, $a_1 = 2(x_1+1) = 2(6+1) = 2\cdot 7$, $a_2 = 2(x_2+1) = 2(\frac{9}{2}+1) = 11$, $a_3 = 2(89+1) = 2\cdot 90 = 2^2\cdot 3^2\cdot 5$. And thus we have $\lambda_1 = 2\cdot 7$, $\lambda_2 = 11$, and then $\lambda_3 = 5$. By Proposition 2.4, we get that $\sqrt{2\cdot 7\epsilon_1} \in k_1 = \mathbb{Q}(\sqrt{7\cdot 5})$, $\sqrt{11\epsilon_2} \in k_2 = \mathbb{Q}(\sqrt{7\cdot 11})$ and $\sqrt{5\epsilon_3} \in k_3 = \mathbb{Q}(\sqrt{5\cdot 11})$. So, $\sqrt{11\epsilon_2}\sqrt{5\epsilon_3} = \sqrt{11\cdot 5}\sqrt{\epsilon_2\epsilon_3} \in K$, as we have $\lambda_2\lambda_3 = 11\cdot 5 = d_3$, then $\sqrt{\epsilon_2\epsilon_3} \in K$.

R e m a r k 3.1. Let $k_3 = \mathbb{Q}(\sqrt{m_1m_2})$ and ϵ_3 be the fundamental unit of k_3 with $N\epsilon_3 = 1$. Let λ_3 be the square-free part of the positive integer $N(\epsilon_3+1)$. Since $\lambda_3 > 1$, λ_3 divides the discriminant of k_3, $\lambda_3 \neq m_1m_2$, and $\sqrt{\lambda_3\epsilon_3} \in k_3 = \mathbb{Q}(\sqrt{m_1m_2})$, so $\sqrt{\epsilon_3} \notin K$. Similarly, we find that $\sqrt{\epsilon_1} \notin K$ and $\sqrt{\epsilon_2} \notin K$.

Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. We know that when we have either $N\epsilon_1 \neq N\epsilon_2 = N\epsilon_3 = 1$, $N\epsilon_2 \neq N\epsilon_1 = N\epsilon_3 = 1$, or $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ so we can have $e_2 = 4$ or $e_2 \neq 4$. In the lemma below, we give $H^1(G_K, E_K)$ the first cohomology group of units of K such that $e_2 \neq 4$, i. e., the prime 2 is not totally ramified in $K/\mathbb{Q}$. We mention here that when $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$, $N\epsilon_1 = N\epsilon_2 = -1 \neq N\epsilon_3 = 1$, $N\epsilon_i \neq N\epsilon_j = N\epsilon_3 = -1$, with $j \neq i = 1, 2$, and $N\epsilon_1 = N\epsilon_2 \neq N\epsilon_3 = -1$, we always have $e_2 \neq 4$.

Lemma 3.1. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that $d_1 = lm_1$ and $d_2 = lm_2$ are square-free integers with $l > 1$ and $gcd(m_1, m_2) = 1$. Then*

1. $H^1(G_K, E_K) \simeq (\mathbb{Z}/2\mathbb{Z})^2$, *when* $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$ *and* $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$.
2. $H^1(G_K, E_K) \simeq (\mathbb{Z}/2\mathbb{Z})^3$, *when*
   - (a) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$ *and* $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$,
   - (b) $N\epsilon_1 = N\epsilon_2 = -1$, $N\epsilon_3 = 1$,
   - (c) $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = -1$, *for* $j \neq k \in \{1, 2\}$,
   - (d) $N\epsilon_1 = N\epsilon_2 = 1$, $N\epsilon_3 = -1$ *and* $\sqrt{\epsilon_1\epsilon_2} \in K$, *or*
   - (e) $N\epsilon_k \neq N\epsilon_j = N\epsilon_3 = 1$ *and* $\sqrt{\epsilon_j\epsilon_3} \in K$, $j \neq k \in \{1, 2\}$ *such that* $e_2 \neq 4$,
   - (f) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ *and* $\sqrt{\epsilon_1\epsilon_2} \in K$ *and* $\sqrt{\epsilon_1\epsilon_3} \in K$ *and* $\sqrt{\epsilon_2\epsilon_3} \in K$ *such that* $e_2 \neq 4$.
3. $H^1(G_K, E_K) \simeq (\mathbb{Z}/2\mathbb{Z})^4$, *when*
   - (a) $N\epsilon_1 = N\epsilon_2 = 1$, $N\epsilon_3 = -1$ *and* $\sqrt{\epsilon_1\epsilon_2} \notin K$,
   - (b) $N\epsilon_k \neq N\epsilon_j = N\epsilon_3 = 1$ *and* $\sqrt{\epsilon_j\epsilon_3} \notin K$, $j \neq k \in \{1, 2\}$ *such that* $e_2 \neq 4$ *or*
   - (c) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ *and* $\sqrt{\epsilon_1\epsilon_2} \in K$, $\sqrt{\epsilon_2\epsilon_3} \in K$, $\sqrt{\epsilon_1\epsilon_3} \in K$, *or* $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ *such that* $e_2 \neq 4$.
4. $H^1(G_K, E_K) \simeq (\mathbb{Z}/2\mathbb{Z})^5$, *when* $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ *and* $\sqrt{\epsilon_1\epsilon_2} \notin K$, $\sqrt{\epsilon_2\epsilon_3} \notin K$, $\sqrt{\epsilon_1\epsilon_3} \notin K$, *and* $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$ *such that* $e_2 \neq 4$.

P r o o f. Recall that λ_1, λ_2 and λ_3 be the square-free part of $N(\epsilon_1+1) = a_1$, $N(\epsilon_2+1) = a_2$ and $N(\epsilon_3+1) = a_3$ respectively, such that $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$. Let $[a_1]$, $[a_2]$, and $[a_3]$ be the class of a_1, a_2 and a_3 in $\mathbb{Q}^*/\mathbb{Q}^{*2}$ respectively, so $[a_1] = [\lambda_1]$, $[a_2] = [\lambda_2]$, and $[a_3] = [\lambda_3]$ where $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$. We know that H is the subgroup of $\mathbb{Q}^*/\mathbb{Q}^{*2}$ generated by the images of d_1, d_2, d_3, a_1, a_2 and a_3 with $d_1 = lm_1$, $d_2 = lm_2$ and $d_3 = m_1m_2$. In the following we study in $\mathbb{Q}^*/\mathbb{Q}^{*2}$ whether $[lm_1]$, $[lm_2]$, $[m_1m_2]$, $[a_1]$, $[a_2]$, and $[a_3]$ are linearly independents. Note that $[m_1m_2]$ belongs to the subgroup generated by $[lm_1]$ and

$[lm_2]$ in $\mathbb{Q}^*/\mathbb{Q}^{*2}$, in other words $[m_1m_2] \in \langle [lm_1], [lm_2] \rangle$. We refer the reader to see the proof of the theorems A, B, C, and D in [6] since in the following we do the same process to give $H^1(G_K, E_K)$.

When $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$, we get that $[a_1] = [a_2] = [a_3] = 1$. And thus, $[lm_1]$ and $[lm_2]$ are linearly independents, i. e., $\langle [lm_1], [lm_2] \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2$. As the three fundamental units with negative norm. Then, by Kubota [11], we have either $E_K = \langle -1, \epsilon_1, \epsilon_2, \epsilon_3 \rangle$ or $E_K = \langle -1, \epsilon_1, \epsilon_2, \sqrt{\epsilon_1\epsilon_2\epsilon_3} \rangle$ is the group of units of K. Thus, we will distinguish the two following cases.

- When $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$, which means that we have $E_K = \langle -1, \epsilon_1, \epsilon_2, \sqrt{\epsilon_1\epsilon_2\epsilon_3} \rangle$. So, by Theorem 2.1, we get that $H^1(G_K, E_K) \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^2$.
- Otherwise, i. e., $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$, then $E_K = \langle -1, \epsilon_1, \epsilon_2, \epsilon_3 \rangle$. On the other hand, we know that $E_{k_1} = \langle -1, \epsilon_1 \rangle$, $E_{k_2} = \langle -1, \epsilon_2 \rangle$ and then $E_{k_3} = \langle -1, \epsilon_3 \rangle$. Therefore, $E_K = E_{k_1}E_{k_2}E_{k_2}$. So, using the Theorem 2.1, we get that $H^1(G_K, E_K) \simeq H \times \mathbb{Z}/2\mathbb{Z} \simeq (\mathbb{Z}/2\mathbb{Z})^3$.

When $N\epsilon_1 = N\epsilon_2 = -1$ and $N\epsilon_3 = 1$, then $[a_1] = [a_2] = 1$. Now we have to check whether $[a_3]$ belongs to the group generated by $[lm_1]$ and $[lm_2]$. By Proposition 2.4 we have $\lambda_3 > 1$ and $\lambda_3 \neq m_1m_2 = d_3$ and then λ_3 divides d_{k_3}. Therefore, we get that $[a_3] = [\lambda_3] \notin \langle [lm_1], [lm_2] \rangle$ and thus $H^1(G_K, E_K) \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^3$.

Assuming $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = -1$ such that $j \neq k = 1, 2$. Then, $[a_k] = [a_3] = 1$. As above, the second assertion, we get that $H^1(G_K, E_K) \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^3$.

When $N\epsilon_1 = N\epsilon_2 = 1$ and $N\epsilon_3 = -1$, so $[a_3] = 1$. Thence, we have to verify whether $[lm_1]$, $[lm_2]$, $[a_1]$, and $[a_2]$ are linearly independents or not. As $N\epsilon_1 = N\epsilon_2 = 1$. Then, we have to distinguish the following cases.

- When $\sqrt{\epsilon_1\epsilon_2} \in K$ (note that we have $E_K = \langle -1, \sqrt{\epsilon_1\epsilon_2}, \epsilon_2, \epsilon_3 \rangle$ see Proposition 2.3). So, according to Proposition 3.1, we have either ($[a_1] = [a_2] = [l]$) or ($[a_1a_2] = [lm_1], [lm_2]$ or $[m_1m_2]$). Note that, we have $\lambda_j > 1$, $\lambda_j \neq lm_j$, and λ_j divides d_{k_j} for $j = 1, 2$. So, we get both $[a_1] = [\lambda_1]$ and $[a_2] = [\lambda_2]$ are not in $\langle [lm_1], [lm_2] \rangle$. Thus, we obtain that $[a_j] \in \langle [lm_1], [lm_2], [a_k] \rangle$ with $j \neq k = 1, 2$. Then, by the Theorem 2.1, we get that $H^1(G_K, E_K) \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^3$.
- Otherwise, i. e., $\sqrt{\epsilon_1\epsilon_2} \notin K$ (note that here we have $E_K = \langle -1, \epsilon_1, \epsilon_2, \epsilon_3 \rangle$), so we have ($[a_1] \neq [l]$ or $[a_2] \neq [l]$) and ($[a_1a_2] \neq [lm_1], [lm_2]$ and $[m_1m_2]$). Hence, $[a_j] \notin \langle [lm_1], [lm_2], [a_k] \rangle$ for $j \neq k = 1, 2$. So, $H^1(G_K, E_K) \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^4$.

Let $N\epsilon_k \neq N\epsilon_j = N\epsilon_3 = 1$, for $j \neq k = 1, 2$ such that $e_2 \neq 4$. Then, $[a_k] = 1$ and thus we have to see whether $[lm_1]$, $[lm_2]$, $[a_j]$, and $[a_3]$ with $j = 1, 2$ are linearly independents. As above, the fourth case, we get that $H^1(G_K, E_K) \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^3$ when $\sqrt{\epsilon_j\epsilon_3} \in K$ with $j = 1, 2$. Otherwise, we get that $H^1(G_K, E_K) \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^4$.

Suppose $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ such that $e_2 \neq 4$. Then, we have to check if $[lm_1]$, $[lm_2]$, $[a_1]$, $[a_2]$ and $[a_3]$ are linearly independents. Therefore, we have to distinguish the following cases.

- When $\sqrt{\epsilon_1\epsilon_2} \in K$. As above, (the fourth case) , we get that $[a_k] \in \langle [lm_1], [lm_2], [a_j] \rangle$ with $j \neq k = 1, 2$. We know that $[a_3] = [\lambda_3] \notin \langle [lm_1], [lm_2] \rangle$. As we are in the case of $\sqrt{\epsilon_1\epsilon_2} \in K$, (i. e., $E_K = \langle -1, \sqrt{\epsilon_1\epsilon_2}, \epsilon_2, \epsilon_3 \rangle$) so $\sqrt{\epsilon_j\epsilon_3} \notin K$ with $j = 1, 2$. Hence, we get that ($[a_j][a_3] = [\lambda_j][\lambda_3] \neq [lm_j]$ and $[m_1m_2]$), and ($[a_j] \neq [m_j]$ or $[a_3] \neq [m_j]$) for $j = 1, 2$. As a result, we have $[a_3] \notin \langle [lm_1], [lm_2], [a_j] \rangle$ and thus $H^1(G_K, E_K) \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^4$.
- When $\sqrt{\epsilon_j\epsilon_3} \in K$ for $j \in \{1, 2\}$, as above, we get that $H^1(G_K, E_K) \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^4$.
- When $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$, (note that we have $E_K = \langle -1, \epsilon_1, \epsilon_2, \sqrt{\epsilon_1\epsilon_2\epsilon_3} \rangle$ see Proposition 2.3). So, we have either ($[a_1][a_2][a_3] = [\lambda_1][\lambda_2][\lambda_3] = [lm_1], [lm_2]$ or $[m_1m_2]$), or ($[a_1][a_2] =$

$[\lambda_1][\lambda_2] = [a_3] = [\lambda_3]$). We know that, $[a_1]$, $[a_2]$ $[a_3] \notin \langle [lm_1], [lm_2] \rangle$. Note that $\sqrt{\epsilon_1\epsilon_3} \notin K$ and $\sqrt{\epsilon_2\epsilon_3} \notin K$ (since $E_K = \langle -1, \epsilon_1, \epsilon_2, \sqrt{\epsilon_1\epsilon_2\epsilon_3} \rangle$), so $[a_3] \notin \langle [lm_1], [lm_2], [a_k] \rangle$ with $k = 1, 2$, but $[a_3] \in \langle [lm_1], [lm_2], [a_1], [a_2] \rangle$. So, $H^1(G_K, E_K) \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^4$.

- Otherwise, i. e., $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$, $\sqrt{\epsilon_1\epsilon_2} \notin K$, $\sqrt{\epsilon_1\epsilon_3} \notin K$, and $\sqrt{\epsilon_2\epsilon_3} \notin K$ in other words $E_K = \langle -1, \epsilon_1, \epsilon_2, \epsilon_3 \rangle$. As a result, we get that $[lm_1]$, $[lm_2]$, $[a_1]$, $[a_2]$ and $[a_3]$ are linearly independents. So, $H^1(G_K, E_K) \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^5$.

When $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $\sqrt{\epsilon_1\epsilon_2} \in K$ and $\sqrt{\epsilon_1\epsilon_3} \in K$ and $\sqrt{\epsilon_2\epsilon_3} \in K$ such that $e_2 \neq 4$ (here we have $E_K = \langle -1, \sqrt{\epsilon_1\epsilon_2}, \sqrt{\epsilon_2\epsilon_3}, \sqrt{\epsilon_1\epsilon_3} \rangle$ see Proposition 2.3). Now we verify if $[lm_1], [lm_2], [a_1], [a_2]$ and $[a_3]$ are linearly independents. We know that when $\sqrt{\epsilon_1\epsilon_2} \in K$, then $[a_k] \in \langle [lm_1], [lm_2], [a_j] \rangle$ with $j \neq k = 1, 2$ and when $\sqrt{\epsilon_j\epsilon_3} \in K$ so $[a_3] \in \langle [lm_1], [lm_2], [a_j] \rangle$, $j = 1, 2$. Thus, $H^1(G_K, E_K) \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^3$. □

In the following we give some examples of $H^1(G_K, E_K)$ such that $e_2 \neq 4$.

E x a m p l e 3.2. In this example we use the same field $K = \mathbb{Q}(\sqrt{7 \cdot 5}, \sqrt{7 \cdot 11})$ of the Example 3.1 (we recall that $e_2 \neq 4$). Since we have $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$, then $H^1(G_K, E_K) \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^4$ or $(\mathbb{Z}/2\mathbb{Z})^5$ (see the lemma above). As we have $\lambda_1 = 2 \cdot 7$, $\lambda_2 = 11$, and then $\lambda_3 = 5$, and $\sqrt{\epsilon_2\epsilon_3} \in K$ (see the Example 3.1), then by the lemma above we get that $H^1(G_K, E_K) \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^4$.

E x a m p l e 3.3. Let $K = \mathbb{Q}(\sqrt{3 \cdot 5 \cdot 7}, \sqrt{3 \cdot 5 \cdot 11})$, where $d_1 = 3 \cdot 5 \cdot 7 = 105$, $d_2 = 3 \cdot 5 \cdot 11 = 165$, $d_3 = 7 \cdot 11 = 77$. Thus, we have $\epsilon_1 = 41 + 4\sqrt{105}$, $\epsilon_2 = \frac{1}{2}(13 + \sqrt{165})$ and then $\epsilon_3 = \frac{1}{2}(9 + \sqrt{77})$ such that $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $e_2 \neq 4$. Hence, we have $a_1 = 2(x_1 + 1) = 2(41 + 1) = 2^2 \cdot 3 \cdot 7$, $a_2 = 2(x_2 + 1) = 2(\frac{13}{2} + 1) = 3 \cdot 5$, $a_3 = 2(x_3 + 1) = 2(\frac{9}{2} + 1) = 11$. We have $\lambda_1 = 3 \cdot 7$, $\lambda_2 = 3 \cdot 5$, and $\lambda_3 = 11$, thus we get that $\sqrt{\epsilon_2\epsilon_3} \in K$ ($\lambda_2\lambda_3 = 3 \cdot 5 \cdot 11 = d_2$). Then $H^1(G_K, E_K) = \langle [3 \cdot 5 \cdot 7], [3 \cdot 5 \cdot 11], [3 \cdot 7], [3 \cdot 5] \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^4$.

3.2. The Pólya groups of the real biquadratic fields $K = \mathbb{Q}(\sqrt{lm_1}, \sqrt{lm_2})$

Theorem 3.1. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that $d_1 = lm_1$ and $d_2 = lm_2$ are square-free integers with $l > 1$ and $gcd(m_1, m_2) = 1$. Let t be the number of the prime divisors of d_K. So,*

1. $\mathcal{P}_O(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-2}$, *when* $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$ *and* $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$.
2. $\mathcal{P}_O(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-3}$, *when*
   (a) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$ *and* $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$,
   (b) $N\epsilon_1 = N\epsilon_2 = -1$, $N\epsilon_3 = 1$,
   (c) $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = -1$, *for* $j \neq k \in \{1, 2\}$,
   (d) $N\epsilon_1 = N\epsilon_2 = 1$, $N\epsilon_3 = -1$ *and* $\sqrt{\epsilon_1\epsilon_2} \in K$, *or*
   (e) $N\epsilon_k \neq N\epsilon_j = N\epsilon_3 = 1$ *and* $\sqrt{\epsilon_j\epsilon_3} \in K$, $j \neq k \in \{1, 2\}$ *such that* $e_2 \neq 4$,
   (f) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ *and* $\sqrt{\epsilon_1\epsilon_2} \in K$, $\sqrt{\epsilon_1\epsilon_3} \in K$, $\sqrt{\epsilon_2\epsilon_3} \in K$ *such that* $e_2 \neq 4$.
3. $\mathcal{P}_O(K) \simeq E_{t-4}$, *when*
   (a) $N\epsilon_1 = N\epsilon_2 = 1$, $N\epsilon_3 = -1$ *and* $\sqrt{\epsilon_1\epsilon_2} \notin K$,
   (b) $N\epsilon_k \neq N\epsilon_j = N\epsilon_3 = 1$ *and* $\sqrt{\epsilon_j\epsilon_3} \notin K$, $j \neq k \in \{1, 2\}$ *such that* $e_2 \neq 4$, *or*
   (c) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ *and* $\sqrt{\epsilon_1\epsilon_2} \in K$, $\sqrt{\epsilon_2\epsilon_3} \in K$, $\sqrt{\epsilon_1\epsilon_3} \in K$ $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ *such that* $e_2 \neq 4$.
4. $\mathcal{P}_O(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-5}$, *when* $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ *and* $\sqrt{\epsilon_1\epsilon_2} \notin K$, $\sqrt{\epsilon_2\epsilon_3} \notin K$, $\sqrt{\epsilon_1\epsilon_3} \notin K$ *and* $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$ *such that* $e_2 \neq 4$.

P r o o f. We have $K = \mathbb{Q}(\sqrt{d}_1, \sqrt{d}_2)$ such that $d_1 = lm_1$ and $d_2 = lm_2$ are square-free integers with $l > 1$ and $gcd(m_1, m_2) = 1$. K is a Galois extension of $\mathbb{Q}$ with $[K : \mathbb{Q}] = 4$ and d_K is the discriminant K. We recall that e_p is the ramification index of a prime number p in $K/\mathbb{Q}$ and thus $e_2 = 1$ when the prime 2 is not ramified in $K/\mathbb{Q}$ and $e_2 = 2$ when the prime 2 is ramified in $K/\mathbb{Q}$. According to Proposition 2.2, we have $|H^1(G_K, E_K)||\mathcal{P}_O(K)| = \prod_{p|d_K} e_p$. Thus, $|\mathcal{P}_O(K)| = \frac{2^t}{|H^1(G_K,E_K)|}$, where t is the number of prime numbers dividing d_K. Thence, we have $\mathcal{P}_O(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-s}$ where s is satisfying $(\mathbb{Z}/2\mathbb{Z})^s \simeq H^1(G_K, E_K)$. By the Lemma 3.1, we have when $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$ and $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$, then $H^1(G_K, E_K) \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Therefore, we get that $\mathcal{P}_O(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-2}$.

Similarly, as above, we deduce the other results of the theorem. □

4. The Pólya Fields of The Real Biquadratic Fields $K = \mathbb{Q}(\sqrt{lm_1}, \sqrt{lm_2})$

In this section, we aim to determine the Pólya fields of the real biquadratic fields of $K = \mathbb{Q}(\sqrt{d}_1, \sqrt{d}_2)$ such that $d_1 = lm_1$ and $d_2 = lm_2$ are square-free integers with $l > 1$ and $gcd(m_1, m_2) = 1$. Let p, p_1, p_2, p_3, p_4 and then p' be the prime integers congruent to 1 (mod 4) and let q, q_1, q_2, q_3, q_4 and then q' be the prime integers congruent to 3 (mod 4).

Since we are going to characterize the Pólya fields of $K = \mathbb{Q}(\sqrt{d}_1, \sqrt{d}_2)$ such that $d_1 = lm_1$ and $d_2 = lm_2$. So, we need the discriminant of K over $\mathbb{Q}$ which determined in [12] and [13] by: $d_K = (lm_1m_2)^2$, when $(d_1, d_2) \equiv (1, 1) \pmod 4$. And when $(d_i, d_j) \equiv (1, 2)$, $(1, 3)$ or $(3, 3) \pmod 4$ with $i \neq j \in \{1, 2\}$, $d_K = (4lm_1m_2)^2$. In the following theorem we give the Pólya fields of K in the cases of $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$, $N\epsilon_1 = N\epsilon_2 = -1 \neq N\epsilon_3 = 1$, and $N\epsilon_i \neq N\epsilon_j = N\epsilon_3 = -1$, with $j \neq i = 1, 2$. Note that in those cases we have $e_2 \neq 4$ and the primes dividing $d_1 = lm_1$ and $d_2 = lm_2$ are not congruent to 3 (mod 4). So, in the theorem below l must be congruent to 1 (mod 4).

Theorem 4.1. *Let $K = \mathbb{Q}(\sqrt{d}_1, \sqrt{d}_2)$ such that $d_1 = lm_1$ and $d_2 = lm_2$ are square-free integers with $l > 1$ and $gcd(m_1, m_2) = 1$ and put $j \neq k \in \{1, 2\}$.*

We assume $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$. Then, K is a Pólya field if and only if one of the following assertions is satisfied:

1. *$d_i = lp_1$ and $d_j = lp_2$, with $l = p$,*
2. *$d_i = lp_1$ and $d_j = 2l$, with $l = p$.*

Now we assume that $N\epsilon_1 = N\epsilon_2 = -1$, $N\epsilon_3 = 1$. So, K is a Pólya field if and only if one of the following assertions is satisfied:

1. *$d_i = lp_1$ and $d_j = lp_2$ where $l = p$,*
2. *$d_i = lp_1$ and $d_j = 2l$ where $l = p$.*

We suppose that $N\epsilon_i \neq N\epsilon_j = N\epsilon_3 = -1$. So, K is a Pólya field if and only if one of the following assertions is satisfied:

1. *$d_i = lp_1$ and $d_j = lp_2$,*
2. *$d_i = lp_1$ and $d_j = 2l$,*
3. *$d_j = lp_1$ and $d_i = 2l$,*

where in the three items above we have $l = p$.

P r o o f. We know that, K is a Pólya field if and only if $\mathcal{P}o(K)$ is trivial. By the Theorem 3.1, we have the following cases.

(i) When $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$ and $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$, then $\mathcal{P}_O(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-2}$ where t is the number of prime divisors of d_K, and thus K is a Pólya field if and only if $t = 2$. Note that this case can not occur because t must be ≥ 3. On the other hand, when $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$, so $\mathcal{P}_O(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-3}$. So, K is a Pólya field if and only if $t = 3$. Hence

- We suppose $(d_i, d_j) \equiv (m_i, m_j) \equiv (1, 1) \pmod 4$, then by Williams [12] we have $d_K = (lm_1m_2)^2$. Thence, K is a Pólya field if and only if $d_i = lp_1$ and $d_j = lp_2$ with $l = p \equiv 1 \pmod 4$.
- Now we suppose $(d_i, d_j) \equiv (m_i, m_j) \equiv (1, 2) \pmod 4$, then $d_K = (4lm_1m_2)^2$. So, K is a Pólya field if and only if $d_i = lp_1$, $d_j = 2l$ with $l = p$.

(ii) When $N\epsilon_1 = N\epsilon_2 = -1$ and $N\epsilon_3 = 1$, then $\mathcal{P}_O(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-3}$. Thus, K is a Pólya field if and only if $t = 3$. When $(d_i, d_j) \equiv (m_i, m_j) \equiv (1, 1) \pmod 4$, we get the item 1, and when $(d_i, d_j) \equiv (m_i, m_j) \equiv (1, 2) \pmod 4$, we have the item 2.

(iii) When $N\epsilon_i \neq N\epsilon_j = N\epsilon_3 = -1$ with $i \neq j \in \{1, 2\}$, then $\mathcal{P}_O(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-3}$. Hence, K is a Pólya field if and only if $t = 3$. When $(d_i, d_j) \equiv (m_i, m_j) \equiv (1, 1) \pmod 4$, we get the item 1 and when $(d_i, d_j) \equiv (m_i, m_j) \equiv (1, 2) \pmod 4$, we obtain 2. And then when $(d_j, d_i) \equiv (m_j, m_i) \equiv (1, 2) \pmod 4$, we have 3. □

In the following theorem, we give the Pólya fields of K in the case of $N\epsilon_1 = N\epsilon_2 \neq N\epsilon_3 = -1$. We mention here that $e_2 \neq 4$.

Theorem 4.2. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that $d_1 = lm_1$ and $d_2 = lm_2$ are square-free integers with $l > 1$ and $gcd(m_1, m_2) = 1$ and put $j \neq k \in \{1, 2\}$. Let $N\epsilon_1 = N\epsilon_2 \neq N\epsilon_3 = -1$.*

We suppose $\sqrt{\epsilon_1\epsilon_2} \in K$. Then, K is a Pólya field if and only if one of the following assertions is satisfied:

1. *$d_i = lp_1$, $d_j = lp_2$, where $l = p$,*
2. *$d_i = lp_1$, $d_j = 2l$, where $l = p$.*

Otherwise, i. e., $\sqrt{\epsilon_1\epsilon_2} \notin K$. Then, K is a Pólya field if and only if one of the following assertions is satisfied:

1. *$d_i = lp_1p_2$, $d_j = lp_3$, where $l = p$,*
2. *$d_i = lp_1p_2$, $d_j = 2l$, where $l = p$,*
3. *$d_i = lp_1$, $d_j = 2lp_2$, where $l = p$,*
4. *$d_i = lp_1$, $d_j = lp_2$, where $l = pp'$,*
5. *$d_i = lp_1$ $d_j = 2l$ where $l = pp'$.*

P r o o f. By the Theorem 3.1, we have the following cases.

(i) When $N\epsilon_1 = N\epsilon_2 \neq N\epsilon_3 = -1$ and $\sqrt{\epsilon_1\epsilon_2} \in K$, then $\mathcal{P}_O(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-3}$ where t is the number of prime divisors of d_K. So, K is a Pólya field if and only if we have either the item 1, or 2.

(ii) When $N\epsilon_1 = N\epsilon_2 \neq N\epsilon_3 = -1$ and $\sqrt{\epsilon_1\epsilon_2} \notin K$. Then, $\mathcal{P}_O(K) \simeq E_{t-4}$. So, K is a Pólya field if and only if $t = 4$.

- When $(d_i, d_j) \equiv (1, 1) \pmod 4$. So, K is a Pólya field if and only if $d_i = lp_1p_2$ $d_j = lp_3$ such that $l = p$. When $l = pp'$, we get the item 4.
- Now when $(d_i, d_j) \equiv (1, 2) \pmod 4$, we get the other items of the theorem. □

Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, $d_1 = lm_1$ and $d_2 = lm_2$ such that $N\epsilon_i \neq N\epsilon_j = N\epsilon_3 = 1$, for $i \neq j \in \{1, 2\}$. Note that in this case we can have either $e_2 = 4$ (since we can have $(d_1, d_2) \equiv (2, 3), (3, 2) \pmod 4$) or $e_2 \neq 4$ (since we can have $(d_1, d_2) \equiv (1, 1), (1, 2), (2, 1), (1, 3), (3, 1) \pmod 4$) note that $(d_1, d_2) \not\equiv (3, 3) \pmod 4$ since $N\epsilon_i \neq N\epsilon_j$, with $i \neq j \in \{1, 2\}$. In the following theorem we give the Pólya fields of K where $e_2 \neq 4$. As we have $N\epsilon_i \neq N\epsilon_j = N\epsilon_3 = 1$, for $i \neq j \in \{1, 2\}$ and l dividing d_1 and d_2, then the divisors of l are $\equiv 1 \pmod 4$.

Theorem 4.3. *Let* $K=\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$ *such that* $d_1=lm_1$ *and* $d_2=lm_2$ *are square-free integers with* $l>1$ *and* $gcd(m_1,m_2)=1$. *Let* $N\epsilon_i\neq N\epsilon_j=N\epsilon_3=1$, *for* $i\neq j\in\{1,2\}$ *such that* $e_2\neq 4$.

Assuming $\sqrt{\epsilon_j\epsilon_3}\in K$, $j=1,2$. *So,* K *is a Pólya field if and only if one of the following assertions is satisfied:*

1. $d_i=lp_1$, $d_j=lp_2$,
2. $d_i=lp_1$, $d_j=2l$,
3. $d_j=lp_1$, $d_i=2l$,

   where in the three items above we have $l=p$.

Otherwise. Then, K *is a Pólya field if and only if one of the following assertions is satisfied:*

1. $d_i=lp_1p_2$ *and* $d_j=lp_3$,
2. $d_i=lp_1$ *and* $d_j=lp_2p_3$ *or* lq_1q_2,
3. $d_i=lp_1p_2$ *and* $d_j=2l$,
4. $d_i=lp_1$ *and* $d_j=2lp_2$ *or* $2lq$,
5. $d_j=lp_1p_2$ *or* lq_1q_2 *and* $d_i=2l$,
6. $d_j=lp_1$ *and* $d_i=2lp_2$,
7. $d_i=lp_1$ *and* $d_j=lq_1$,

   where in the items above we have $l=p$,
8. $d_i=lp_1$ *and* $d_j=lp_2$,
9. $d_i=lp_1$ *and* $d_j=2l$,
10. $d_j=lp_1$ *and* $d_i=2l$,

    such that $l=pp'$.

P r o o f. We know that, when $N\epsilon_i\neq N\epsilon_j=N\epsilon_3=1$ such that $e_2\neq 4$, for $i\neq j\in\{1,2\}$, then we have $(d_i,d_j)\equiv(1,1),(1,2),(2,1),(1,3)\pmod 4$. By Theorem 3.1, we have the following cases.

(i) When $N\epsilon_i\neq N\epsilon_j=N\epsilon_3=1$, and $\sqrt{\epsilon_j\epsilon_3}\in K$ such that $e_2\neq 4$ with $i\neq j\in\{1,2\}$. Then, $\mathcal{P}_O(K)\simeq(\mathbb{Z}/2\mathbb{Z})^{t-3}$. So, K is a Pólya field if and only if $t=3$. Therefore, when $(d_i,d_j)\equiv(1,1)\pmod 4$ we get the item 1 and when $(d_i,d_j)\equiv(1,2)\pmod 4$ we have the item 2, lastly when $(d_j,d_i)\equiv(1,2)\pmod 4$ we obtain the item 3.

(ii) When $N\epsilon_i\neq N\epsilon_j=N\epsilon_3=1$ and $\sqrt{\epsilon_j\epsilon_3}\notin K$ such that $e_2\neq 4$ with $i\neq j\in\{1,2\}$. Then, $\mathcal{P}_O(K)\simeq(\mathbb{Z}/2\mathbb{Z})^{t-4}$. Thence, K is a Pólya field if and only if $t=4$.

- When $(d_i,d_j)\equiv(1,1)\pmod 4$, then $d_K=(lm_1m_2)^2$. When $l=p$, so K is a Pólya field if and only if either $d_i=lp_1p_2$, $d_j=lp_3$ or the item 2. When $l=pp'$, we have the item 8.
- We suppose that $(d_i,d_j)\equiv(1,2)\pmod 4$. If $l=p$ thus K is a Pólya field if and only if we have either the item 3, or 4. When $l=pp'$, we get the item 9.
- When $(d_j,d_i)\equiv(1,2)\pmod 4$. When $l=p$, we have either the item 5, or 6. And when $l=pp'$, we obtain the item 10.
- Lastly, when $(d_i,d_j)\equiv(1,3)\pmod 4$, we get the item 7. □

Consider $K=\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$, $d_1=lm_1$ and $d_2=lm_2$ such that $N\epsilon_1=N\epsilon_2=N\epsilon_3=1$. Under the condition of the norm, we can have either $e_2\neq 4$ or $e_2=4$. In the following theorem we give the Pólya fields of K where $N\epsilon_1=N\epsilon_2=N\epsilon_3=1$ and $\sqrt{\epsilon_1\epsilon_2}\in K$ and $\sqrt{\epsilon_2\epsilon_3}\in K$ and $\sqrt{\epsilon_1\epsilon_3}\in K$ (i. e., $E_K=\langle -1,\sqrt{\epsilon_1\epsilon_2},\sqrt{\epsilon_2\epsilon_3},\sqrt{\epsilon_1\epsilon_3}\rangle$) such that $e_2\neq 4$. As we have $N\epsilon_1=N\epsilon_2=N\epsilon_3=1$, so l can be $\equiv 1\pmod 4$ or $\equiv 3\pmod 4$.

Theorem 4.4. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that $d_1 = lm_1$ and $d_2 = lm_2$ are square-free integers with $l > 1$ and $gcd(m_1, m_2) = 1$. Let $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $\sqrt{\epsilon_1\epsilon_2} \in K$ and $\sqrt{\epsilon_2\epsilon_3} \in K$ and $\sqrt{\epsilon_1\epsilon_3} \in K$ such that $e_2 \neq 4$. Then, K is a Pólya field if and only if one of the following assertions is satisfied:*

1. $d_i = lp_1$, $d_j = lp_2$, *where* $l = p$,
2. $d_i = lq_1$, $d_j = lq_2$, *where* $l = q$,
3. $d_i = lp_1$, $d_j = 2l$, *where* $l = p$,
4. $d_j = lq_1$, $d_i = 2l$, *with* $l = q$.

P r o o f. As we have $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $\sqrt{\epsilon_1\epsilon_2} \in K$ and $\sqrt{\epsilon_2\epsilon_3} \in K$ and $\sqrt{\epsilon_1\epsilon_3} \in K$ such that $e_2 \neq 4$, so by Theorem 3.1 we have $\mathcal{P}_O(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-3}$. Hence, K is a Pólya field if and only if $t = 3$. Therefore, when $(d_i, d_j) \equiv (1, 1) \pmod 4$, we know that $d_K = (lm_1m_2)^2$ and thus we get the items 1, 2. And when $(d_i, d_j) \equiv (1, 2) \pmod 4$, we have $d_K = (4lm_1m_2)^2$ and thus we get the items 3, 4. When $(d_i, d_j) \equiv (1, 3)$ or $(3, 3) \pmod 4$, $i \neq j = 1, 2$ so $d_K = (4lm_1m_2)^2$ and since $t = 3$ we find that these cases can not occur. □

Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, $d_1 = lm_1$ and $d_2 = lm_2$. In the two following theorems, we give the Pólya fields of K such that $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $e_2 \neq 4$. We recall that since $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$, so l can be $\equiv 1 \pmod 4$ or $\equiv 3 \pmod 4$.

Theorem 4.5. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that $d_1 = lm_1$ and $d_2 = lm_2$ are square-free integers with $l > 1$ and $gcd(m_1, m_2) = 1$ and put $j \neq k \in \{1, 2\}$. Let $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $\sqrt{\epsilon_1\epsilon_2} \in K$, $\sqrt{\epsilon_2\epsilon_3} \in K$, $\sqrt{\epsilon_1\epsilon_3} \in K$ or $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ such that $e_2 \neq 4$. Then, K is a Pólya field if and only if one of the following assertions is satisfied:*

1. $d_i = lp_1p_2$ *or* lq_1q_2 *and* $d_j = lp_3$,
2. $d_i = lp_1p_2$ *or* lq_1q_2 *and* $d_j = 2l$,
3. $d_i = lp_1$ *and* $d_j = 2lp_2$ *or* $2lq$,
4. $d_i = lp_1$ *and* $d_j = lq_1$,
5. $d_i = lq_1$ *and* $d_j = lq_2$,
   where in the items above $l = p$,
6. $d_i = lp_1$ *and* $d_j = lp_2$,
7. $d_i = lp_1$ *and* $d_j = 2l$,
   where $l = pp'$.
8. $d_i = lq_1$ *and* $d_j = lpq_2$,
9. $d_i = lq_1$ *and* $d_j = lp_1$,
10. $d_i = lp_1q_1$ *and* $d_j = 2l$,
11. $d_i = lq_1$ *and* $d_j = 2lp$ *or* $2lq_2$,
12. $d_i = lp_1$ *and* $d_j = lp_2$,
    such that $l = q$,
13. $d_i = lp_1$ *and* $d_j = lp_2$,
14. $d_i = lp_1$ *and* $d_j = 2l$,
    where $l = qq'$,
15. $d_i = lq_1$ *and* $d_j = lq_2$,
16. $d_i = lq_1$ *and* $d_j = 2l$,
    where $l = pq$.

P r o o f. According to the Theorem 3.1, we have when $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $\sqrt{\epsilon_1\epsilon_2} \in K$, $\sqrt{\epsilon_2\epsilon_3} \in K$, $\sqrt{\epsilon_1\epsilon_3} \in K$ or $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ such that $e_2 \neq 4$. Then, $\mathcal{P}_O(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-4}$. Hence, K is a Pólya field if and only if $t = 4$.

We suppose that $(d_i, d_j) \equiv (m_i, m_j) \equiv (1,1) \pmod 4$, then we have $d_K = (lm_1m_2)^2$. When $l = p$, then K is a Pólya field if and only if either $d_i = lp_1p_2$ or lq_1q_2 and $d_j = lp_3$. When $l = pp'$, then $d_i = lp_1$, $d_j = lp_2$. If $l = qq'$, so $d_i = lp_1$, $d_j = lp_2$.

When $(d_i, d_j) \equiv (1,1) \pmod 4$, $(m_i, m_j) \equiv (3,3) \pmod 4$. So, we get that $d_i = lq_1$, $d_j = lpq_2$ such that $l = q$. When $l = pq$, so we have $d_i = lq_1$, $d_j = lq_2$.

Assuming $(d_i, d_j) \equiv (1,2) \pmod 4$, then $d_K = (4lm_1m_2)^2$.

- When $l = p$ and $m_j = 2$, then $d_i = lp_1p_2$ or lq_1q_2, $d_j = 2l$. Now for $m_j = 2p_2, 2q_2$ so $d_i = lp_1$, $d_j = 2lp_2, 2lq$.
- We assume $l = pp'$, so $d_i = lp_1$, $d_j = 2l$. When $l = qq'$, we get $d_i = lp_1$, $d_j = 2l$. And when $l = pq$, we obtain $d_i = lq_1$, $d_j = 2l$.
- When $l = q$ and $m_j = 2$, then $d_i = lp_1q_1$, $d_j = 2l$. For $m_j = 2p, 2q_2$, so $d_i = lq_1$, $d_j = 2lp, 2lq_2$.

We suppose that $(d_i, d_j) \equiv (1,3) \pmod 4$, then $d_K = (4lm_1m_2)^2$. For $l = p$, then $d_i = lp_1$ $d_j = lq_1$. When $l = q$, so $d_i = lq_1$ $d_j = lp_1$.

When $(d_i, d_j) \equiv (3,3) \pmod 4$, $i \neq j \in \{1,2\}$ then $d_K = (4lm_1m_2)^2$. When $l = p$, thus $d_i = lq_1$ $d_j = lq_2$. If $l = q$, so $d_i = lp_1$, $d_j = lp_2$. As we have $e_2 \neq 4$, then (d_i, d_j) not congruent to $(2,3) \pmod 4$ for $i \neq j = 1,2$. □

E x a m p l e 4.1. Let $K = \mathbb{Q}(\sqrt{7\cdot 5}, \sqrt{7\cdot 11})$. We have $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$, and $\sqrt{\epsilon_2\epsilon_3} \in K$ and $e_2 \neq 4$ (se Example 3.1). We have $l = 7 \equiv 3 \pmod 4$ and $5 \equiv 1 \pmod 4$ and $11 \equiv 3 \pmod 4$. So by the item 9 of the theorem above, we get that K is a Pólya field.

Theorem 4.6. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that $d_1 = lm_1$ and $d_2 = lm_2$ are square-free integers with $l > 1$ and $gcd(m_1, m_2) = 1$ and put $j \neq k \in \{1,2\}$. Assuming $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $\sqrt{\epsilon_1\epsilon_2} \notin K$, $\sqrt{\epsilon_2\epsilon_3} \notin K$, $\sqrt{\epsilon_1\epsilon_3} \notin K$ and $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$ such that $e_2 \neq 4$. Then, K is a Pólya field if and only if one of the following assertions is satisfied:*

1. $d_i = lp_1p_2$ *or* lq_1q_2 *and* $d_j = lp_3p_4$,
2. $d_i = lp_1p_2$ *or* lq_1q_2 *and* $d_j = lq_3q_4$,
3. $d_i = lp_1p_2p_3$ *or* $lq_1q_2p_1$ *and* $d_j = lp'_1$,
4. $d_i = lp_1p_2p_3$ *or* $lq_1q_2p_1$ *and* $d_j = 2l$,
5. $d_i = lp_1p_3$ *or* lq_1q_3 *and* $d_j = 2lp_2$,
6. $d_i = lp_1p_3$ *or* lq_1q_3 *and* $d_j = 2lq_2$,
7. $d_i = lp_3$ *and* $d_j = 2lp_1p_2, 2lp_1q_1, 2lq_1q_2$,
8. $d_i = lp_1q_1$ *and* $d_j = lq_2$,
9. $d_i = lq_1q_2$ *and* $d_j = lq_3$,
10. $d_i = lp_1p_2$ *and* $d_j = lq_1$,
11. $d_i = lp_1$ *and* $d_j = lp_2q_1$,

   where in the items above we have $l = p$,
12. $d_i = lp_1q_1$ *and* $d_j = lp_2q_2$,
13. $d_i = lq_1$ *and* $d_j = lp_1p_2q_2, lq_2q_3q_4$,
14. $d_i = lp_1p_2$ *and* $d_j = lp_3$,
15. $d_i = lq_1q_2$ *and* $d_j = lp_1$,
16. $d_i = lq_1p_1$ *and* $d_j = lp_2$,

17. $d_i = lq_1$ *and* $d_j = lp_1p_2,\, lq_1q_2$,
18. $d_i = lp_1p_2q_1,\, lq_1q_2q_3$ *and* $d_j = 2l$,
19. $d_i = lp_1q_1$ *and* $d_j = 2lp_2,\, 2lq_2$,
20. $d_i = lq_1$ *and* $d_j = 2lp_1p_2,\, 2lp_1q_2,\, 2lq_2q_3$,
    where $l = q$,
21. $d_i = lq_1$ *and* $d_j = lq_2$,
22. $d_i = lp_1$ *and* $d_j = lq_1$,
23. $d_i = lp_1$ *and* $d_j = lp_2p_3,\, lq_1q_2$,
24. $d_i = lp_1$ *and* $d_j = 2lp_2,\, 2lq_1$,
25. $d_i = lp_1p_2,\, lq_1q_2$ *and* $d_j = 2l$,
    where $l = pp'$,
26. $d_i = lp_1$ *and* $d_j = lp_2p_3,\, lq_1q_2$,
27. $d_i = lp_1$ *and* $d_j = 2lp_2,\, 2lq_1$,
28. $d_i = lp_1p_2,\, lq_1q_2$ *and* $d_j = 2l$,
29. $d_i = lq_1$ *and* $d_j = lq_2$,
30. $d_i = lp_1$ *and* $d_j = lq_1$,
    where $l = qq'$,
31. $d_i = lp_1q_1$ *and* $d_j = lq_2$,
32. $d_i = lp_1$ *and* $d_j = lp_2$,
33. $d_i = lq_1$ *and* $d_j = lp_1$,
34. $d_i = lp_1q_1$ *and* $d_j = 2l$,
35. $d_i = lq_1$ *and* $d_j = 2lq_2,\, 2lp_1$,
    such that $l = pq$,
36. $d_i = lp_1$ *and* $d_j = lp_2$,
37. $d_i = lp_1$ *and* $d_j = 2l$,
    such that $l = pp'p'_1$,
38. $d_i = lp_1$ *and* $d_j = lp_2$,
39. $d_i = lp_1$ *and* $d_j = 2l$,
    where $l = qq'p$,
40. $d_i = lq_1$ *and* $d_j = lq_2$,
41. $d_i = lq_1$ *and* $d_j = 2l$,
    where $l = pp'q$ *or* $qq'q'_1$.

P r o o f. According to the Theorem 3.1, we have $\mathcal{P}_O(K) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-5}$. Thence, K is a Pólya field if and only if $t = 5$.

We suppose $(d_i, d_j) \equiv (m_i, m_j) \equiv (1, 1) \pmod 4$.

- When $l = p$. Hence, we have either $d_i = lp_1p_2,\, lq_1q_2$, $d_j = lp_3p_4$ or $d_i = lp_1p_2,\, lq_1q_2$, $d_j = lq_3q_4$ or $d_i = lp_1p_2p_3,\, lq_1q_2p_1$, $d_j = lp'_1$.
- If $l = pp'$, then $d_i = lp_1$, $d_j = lp_2p_3,\, lq_1q_2$.
- When $l = qq'$, so $d_i = lp_1$, $d_j = lp_2p_3,\, lq_1q_2$.
- If $l = pp'p'_1$, therefore $d_i = lp_1$, $d_j = lp_2$.
- Now for $l = qq'p$, thus $d_i = lp_1$, $d_j = lp_2$.

When $(d_i, d_j) \equiv (1, 1) \pmod 4$, $(m_i, m_j) \equiv (3, 3) \pmod 4$.

• When $l = q$, so we get either $d_i = lp_1q_1$, $d_j = lp_2q_2$ or $d_i = lq_1$, $d_j = lp_1p_2q_2, lq_2q_3q_4$. If $l = pq$, then we have $d_i = lp_1q_1$, $d_j = lq_2$. If $l = pp'q$ or $qq'q'_1$, we get that $d_i = lq_1$, $d_j = lq_2$.

Assuming $(d_i, d_j) \equiv (1, 2) \pmod 4$, then $d_K = (4lm_1m_2)^2$.

• When $l = p$ and $m_j = 2$. So, K is a Pólya field if and only if $d_i = lp_1p_2p_3, lq_1q_2p_1$, $d_j = 2l$. For $m_j = 2p_2, 2q_2$ we get either $d_i = lp_1p_3, lq_1q_3$, $d_j = 2lp_2$ or $d_i = lp_1p_3, lq_1q_3$, $d_j = 2lq_2$. For $m_j = 2p_1p_2, 2p_1q_1, 2q_1q_2$, we obtain $d_i = lp_3$, $d_j = 2lp_1p_2, 2lp_1q_1, 2lq_1q_2$.

• We assume $l = pp'$, then we get that either $d_i = lp_1$, $d_j = 2lp_2, 2lq_1$ or $d_i = lp_1p_2, lq_1q_2$, $d_j = 2l$.

• When $l = qq'$, then we get either $d_i = lp_1$, $d_j = 2lp_2, 2lq_1$ or $d_i = lp_1p_2, lq_1q_2$, $d_j = 2l$.

• If $l = pp'p'_1$, then $d_i = lp_1$, $d_j = 2l$.

• When $l = qq'p$, thence, $d_i = lp_1$, $d_j = 2l$.

When $l = q$ and $m_j = 2$, so $d_i = lp_1p_2q_1, lq_1q_2q_3$, $d_j = 2l$. For $m_j = 2p_2, 2q_2$, we get that $d_i = lp_1q_1$ and $d_j = 2lp_2, 2lq_2$. For $m_j = 2p_1p_2, 2p_1q_1, 2q_1q_2$ so $d_i = lq_1$ and $d_j = 2lp_1p_2, 2lp_1q_2, 2lq_2q_3$.

• We assume $l = pq$, then we get that either $d_i = lp_1q_1$, $d_j = 2l$ or $d_i = lq_1$, $d_j = 2lq_2, 2lp_1$.

• When $l = qpp'$ or $qq'q'_1$, so $d_i = lq_1$, $d_j = 2l$.

We suppose that $(d_i, d_j) \equiv (3, 3) \pmod 4$, then $d_K = (4lm_1m_2)^2$.

• When $l = p$, thence $d_i = lp_1q_1$, $d_j = lq_2$. When $l = pp'$, we get $d_i = lq_1$, $d_j = lq_2$.

• For $l = q$, so $d_i = lp_1p_2$, $d_j = lp_3$ or $d_i = lq_1q_2$, $d_j = lp_1$. If $l = qq'$, then we get $d_i = lq_1$, $d_j = lq_2$.

• When $l = pq$, we get that $d_i = lp_1$, $d_j = lp_2$.

We assume that $(d_i, d_j) \equiv (1, 3) \pmod 4$. So, $d_K = (4lm_1m_2)^2$.

• We put $l = p$, thus we have either $d_i = lq_1q_2$, $d_j = lq_3$ or $d_i = lp_1p_2$, $d_j = lq_1$ or $d_i = lp_1$, $d_j = lp_2q_1$. When $l = pp'$, we get $d_i = lp_1$, $d_j = lq_1$.

• We let $l = q$, so $d_i = lq_1p_1$, $d_j = lp_2$ or $d_i = lq_1$, $d_j = lp_1p_2, lq_1q_2$. If $l = qq'$, then we get $d_i = lp_1$, $d_j = lq_1$.

• When $l = pq$, then $d_i = lq_1$, $d_j = lp_1$. □

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