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Star product and star function

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Звездочное умножение и звездочные функции

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Abstract. We give a brief review on star products and star functions [8, 9]. We introduce a star product on polynomials. Extending the product to functions on complex space, we introduce exponential element in the star product algebra. By means of the star exponential functions we can define several functions called star functions in the algebra. We show certain examples.

Keywords: Moyal product; star product; star product algebra; star exponential functions

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Аннотация. Мы даем короткий обзор звездочных умножений и звездочных функций, см. [8, 9]. Сначала мы вводим звездочное умножение для многочленов. Затем, распространяя произведение на функции, заданные на комплексном пространстве, мы вводим экспоненты в алгебрах с звездочным умножением. С помощью звездочно показательных функций мы можем определить некоторые функции в этих алгебрах, называемые звездочными функциями. Мы также указываем некоторые примеры.

Ключевые слова: умножение Мойяла; звездочное умножение; алгебры со звездочным умножением; звездочно показательные функции

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1. Star product on polynomials

1.1. Moyal product

The Moyal product is a well-known example of star product [2,3].

For polynomials f, g of the variables $(u_1, \dots, u_m, v_1, \dots, v_m)$, the Moyal product $f *_{O} g$ is given by the power series of the bidifferential operators $\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}$ such that

$$\begin{aligned} f *_{O} g &= f \exp \frac{i\hbar}{2} \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right) \ g = f \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2} \right)^k \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right)^k g \\ &= fg + \frac{i\hbar}{2} f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right) g + \frac{1}{2!} \left(\frac{i\hbar}{2} \right)^2 f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right)^2 g \\ &\quad + \cdots + \frac{1}{k!} \left(\frac{i\hbar}{2} \right)^k f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right)^k g + \cdots \end{aligned} \tag{1.1.1}$$

where $\hbar$ is a positive number and the overleft arrow $\overleftarrow{\partial}$ means that the partial derivative is acting on the polynomial on the left and the overright arrow similar, for example

$$f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right) g = \sum_{j=1}^{m} \left(\partial_{v_j} f \ \partial_{u_j} g - \partial_{u_j} f \ \partial_{v_j} g \right).$$

Although the Moyal product is defined as a formal power series of bidifferential operators, this becomes a finite sum on polynomials.

P r o p o s i t i o n 1.1.1. *The Moyal product is well-defined on polynomials, and associative.*

Other typical star products are normal product $*_{N}$, anti-normal product $*_{A}$ given similarly by

$$f *_{N} g = f \exp i\hbar \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} \right) \ g, \quad f *_{A} g = f \exp \left\{ -i\hbar \left(\overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right) \right\} \ g.$$

These are also well-defined on polynomials and associative.

By direct calculation we see easily

P r o p o s i t i o n 1.1.2.

(i) *For these star products, the generators $(u_1, \dots, u_m, v_1, \dots, v_m)$ satisfy the canonical commutation relations*

$$[u_k, v_l]_{*_L} = -i\hbar\delta_{kl}, \ [u_k, u_l]_{*_L} = [v_k, v_l]_{*_L} = 0, \quad (k, l = 1, 2, \dots, m)$$

where $*_{L}$ stands for $*_{O}$, $*_{N}$ and $*_{A}$.

(ii) *Then the algebras $(\mathbb{C}[u, v], *_L)$ $(L = O, N, A)$ are mutually isomorphic and isomorphic to the Weyl algebra.*

Actually the algebra isomorphism

$$I_O^N : (\mathbb{C}[u,v], *_O) \to (\mathbb{C}[u,v], *_N)$$

is given explicitly by the power series of the differential operator such as

$$I_N^O\,(f) = \exp\left(-\frac{i\hbar}{2}\partial_u\partial_v\right)(f) = \sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{i\hbar}{2}\right)^l(\partial_u\partial_v)^l\,(f). \tag{1.1.2}$$

And other isomorphisms are given in the similar form.

R e m a r k 1.1.1. We remark here that these facts are well-known as ordering problem in physics [1].

1.2. Star product

Now we define a star product on complex domain by generalizing the previous products.

Let Λ be an arbitrary $n \times n$ complex matrix. We consider a bidifferential operator

$$\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w} = (\overleftarrow{\partial_{w_1}}, \cdots, \overleftarrow{\partial_{w_n}})\Lambda(\overrightarrow{\partial_{w_1}}, \cdots, \overrightarrow{\partial_{w_n}}) = \sum_{k,l=1}^{n}\Lambda_{kl}\overleftarrow{\partial_{w_k}}\overrightarrow{\partial_{w_l}} \tag{1.2.3}$$

where $(w_1, \cdots, w_n)$ is a generators of polynomials.

Now we define a star product similar to (1) by

D e f i n i t i o n 1.2.1.

$$f *_{\Lambda} g = f \exp\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right)\; g. \tag{1.2.4}$$

R e m a r k 1.2.1. [9]

(i) The star product $*_{\Lambda}$ is a generalization of the previous products. Actually

- if we put $\Lambda = \begin{pmatrix} 0 & -1_m \\ 1_m & 0 \end{pmatrix}$ then we have the Moyal product
- if $\Lambda = 2\begin{pmatrix} 0 & 0 \\ 1_m & 0 \end{pmatrix}$, then we have the normal product
- if $\Lambda = 2\begin{pmatrix} 0 & -1_m \\ 0 & 0 \end{pmatrix}$ then we have the anti-normal product

(ii) If Λ is a symmetric matrix, the star product $*_{\Lambda}$ is commutative.

Then similarly as before we see easily

Theorem 1.2.1. *For an arbitrary Λ, the star product $*_{\Lambda}$ is well-defined on polynomials, and associative.*

1.3. Equivalence and geometric picture of Weyl algebra

In this section, we take Λ as a special class of matrices in order to represent Weyl algebra, cf. [4, 7]. We consider the following complex matrices Λ :

$$\Lambda = J + K,$$

where K is an arbitrary $2m \times 2m$ complex symmetric matrix and

$$J = \begin{pmatrix} 0 & -1_m \\ 1_m & 0 \end{pmatrix}.$$

Since Λ is determined by K, we denote the star product by $*_{_K}$ instead of $*_{_\Lambda}$.

We consider polynomials in variables $(w_1, \cdots, w_{2m}) = (u_1, \cdots, u_m, v_1, \cdots, v_m)$. By a easy calculation one obtains

P r o p o s i t i o n 1.3.1.

(i) *For a star product* $*_{_K}$, *the generators* $(u_1, \ldots, u_m, v_1, \ldots, v_m)$ *satisfy the canonical commutation relations*

$$[u_k, v_l]_{*_K} = -i\hbar\delta_{kl}, \quad [u_k, u_l]_{*_K} = [v_k, v_l]_{*_K} = 0, \quad (k, l = 1, 2, \ldots, m).$$

(ii) *Then the algebra* $(\mathbb{C}[u, v], *_{_K})$ *is isomorphic to the Weyl algebra, and the algebra is regarded as a polynomial representation of the Weyl algebra.*

Equialence As in the case of typical star products, we have algebra isomorphisms as follows.

P r o p o s i t i o n 1.3.2. *For arbitrary star product algebras* $(\mathbb{C}[u, v], *_{_{K_1}})$ *and* $(\mathbb{C}[u, v], *_{_{K_2}})$ *we have an algebra isomorphism* $I_{K_1}^{K_2} : (\mathbb{C}[u, v], *_{_{K_1}}) \to (\mathbb{C}[u, v], *_{_{K_2}})$ *given by the power series of the differential operator* $\partial_w(K_2 - K_1)\partial_w$ *such that*

$$I_{K_1}^{K_2}(f) = \exp\left(\frac{i\hbar}{4}\, \partial_w(K_2 - K_1)\partial_w\right)(f),$$

where $\partial_w(K_2 - K_1)\partial_w = \sum_{kl}(K_2 - K_1)_{kl}\partial_{w_k}\partial_{w_l}$.

R e m a r k 1.3.1.

1. By the previous proposition we see the algebras $(\mathbb{C}[u, v], *_{_K})$ are mutually isomorphic and isomorphic to the Weyl algebra. Hence we have a family of star product algebras $\{(\mathbb{C}[u, v], *_{_K})\}_K$ where each element is regarded as a polynomial representation of the Weyl algebra.

2. The above equivalences are also possible to make for star products $*_{_\Lambda}$ for arbitrary Λ's with a common skew symmetric part.

By a direct calculation we have

Theorem 1.3.1. *Isomorphisms satisfy the following chain rule:*

1. $I^{K_1}_{K_3} I^{K_3}_{K_2} I^{K_2}_{K_1} = Id\,, \quad \forall K_1, K_2, K_3$
2. $\left(I^{K_2}_{K_1}\right)^{-1} = I^{K_1}_{K_2}\,, \quad \forall K_1, K_2$

According to the previous theorem, we introduce an infinite dimensional bundle and a connection over it and using parallel sections of this bundle we have a geometric picture for the family of the star product algebras $\{(\mathbb{C}[u,v], *_{K})\}_K$.

Algebra bundle We set $\mathcal{S} = \{K\}$ the space of all $2m \times 2m$ symmetric complex matrices. We consider a trivial bundle over $\mathcal{S}$ whose fibers are the star product algebras

$$\pi : E = \prod_{K \in \mathcal{S}} (\mathbb{C}[u,v], *_{K}) \to \mathcal{S}, \quad \pi^{-1}(K) = (\mathbb{C}[u,v], *_{K}).$$

Then the previous proposition shows that fibers $(\mathbb{C}[u,v], *_{K})$ are mutually isomorphic and are isomorphic to the Weyl algebra, and the isomorphisms $I^{K_2}_{K_1}$ give an isomorphism between fibers.

Connection and parallel sections For a curve $C : K = K(t)$ in the base space $\mathcal{S}$, starting from $K(0) = K$, we define a parallel translation of a polynomial $f \in (\mathbb{C}[u,v], *_{K})$ by

$$f(t) = \exp \frac{i\hbar}{4}\, \partial_w (K(t) - K) \partial_w (f).$$

It is easy to see $f(0) = f$. By differentiating the parallel translation we have a connection of this bundle such that

$$\nabla_X f(K) = \frac{d}{dt} f(t)|_{t=0}(K) = \frac{i\hbar}{4}\, \partial_w X \partial_w\ f(K)|_{t=0}, \quad X = \dot{K}(t)|_{t=0},$$

where $f(K)$ is a smooth section of the bundle E.

We set $\mathcal{P}$ the space of all parallel sections of this bundle. Since $I^{K_2}_{K_1}$ are algebra isomorphisms

$$I^{K_2}_{K_1}(f(K_1) *_{K_1} g(K_1)) = \left(I^{K_2}_{K_1}(f(K_1)\right) *_{K_2} \left(I^{K_2}_{K_1}(g(K_1)\right),$$

we have a star product on the space of parallel sections $f, g \in \mathcal{P}$ by

$$f * g(K) = f(K) *_{K} g(K).$$

Then we have

Theorem 1.3.2.

(i) *The space of the parallel sections* $\mathcal{P}$ *consists of the sections such that*

$$\nabla_X f = \frac{i\hbar}{4}\, \partial_w X \partial_w\ f = 0, \quad \forall X.$$

(ii) *The space* $\mathcal{P}$ *is canonically equipped with the star product* $*$, *and the associative algebra* $(\mathcal{P}, *)$ *is isomorphic to the Weyl algebra.*

R e m a r k 1.3.2. The algebra $(\mathcal{P}, *)$ is regarded as a geometric realization of the Weyl algebra.

2. Extension to functions

We want to extend the star products $*_{_\Lambda}$ for an arbitrary complex matrix Λ from polynomials to functions, cf. [6].

2.1. Star product on certain holomorphic function space

We want to transfer the star products $*_{_\Lambda}$ from polynomials to functions. However, the product is not necessarily convergent for ordinary smooth functions, hence we need to restrict the product to certain subset of smooth functions.

There may be many such spaces. In this note we consider the following space of certain entire functions.

Semi-norm Let $f(w)$ be a holomorphic function on $\mathbb{C}^n$. For a positive number p, we consider a family of semi-norms $\{|\cdot|_{p,s}\}_{s>0}$ given by

$$|f|_{p,s} = \sup_{w\in\mathbb{C}^n} |f(w)| \exp(-s|w|^p), \quad |w| = \sqrt{|w_1|^2 + \cdots + |w_n|^2}.$$

Space We put

$$\mathcal{E}_p = \{f : \text{ entire} \mid |f|_{p,s} < \infty, \forall s > 0\}.$$

With the semi-norms the space $\mathcal{E}_p$ becomes a Fréchet space. As to the star products, we have for any matrix Λ.

Theorem 2.1.1.

(i) *For* $0 < p \leqslant 2$, $(\mathcal{E}_p, *_{_\Lambda})$ *is a Frechét algebra. That is, the product converges for any elements, and the product is continuous with respect to this topology.*

(ii) *Moreover, for any* Λ' *with the common skew symmetric part with* Λ, *the map*

$$I_{_\Lambda}^{\Lambda'} = \exp\left(\frac{i\hbar}{4}\, \partial_w(\Lambda' - \Lambda)\partial_w\right)$$

is a well-defined algebra isomorphism from $(\mathcal{E}_p, *_{_\Lambda})$ *to* $(\mathcal{E}_p, *_{_{\Lambda'}})$. *That is, the expansion convergies for every element, and the operator is continuous with respect to this topology.*

(iii) *For* $p > 2$, *the multiplication* $*_\Lambda : \mathcal{E}_p \times \mathcal{E}_q \to \mathcal{E}_p$ *is a well-defined for* q *such that* $(1/p) + (1/q) = 2$, *and* $(\mathcal{E}_p, *_\Lambda)$ *is a* $\mathcal{E}_q$*-bimodule.*

3. Star exponentials

Since we have a complete topological algebra, we can consider exponential elements in the star product algebra $(\mathcal{E}_p, *_\Lambda)$, cf. [9].

3.1. Definition

For a polynomial H_*, we want to define a star exponential $\exp_*(tH_*/i\hbar)$. However, except special cases, the expansion

$$\sum_n \frac{t^n}{n!}\left(\frac{H_*}{i\hbar}\right)^n$$

is not convergent, so we define a star exponential by means of a differential equation.

D e f i n i t i o n 3.1.1. The star exponential $\exp_*(tH_*/i\hbar)$ is given as a solution of the following differential equation

$$\frac{d}{dt}F_t = H_* *_\Lambda F_t, \quad F_0 = 1. \tag{3.1.1}$$

3.2. Examples

We are interested in the star exponentials of linear, and quadratic polynomials. For these, we can solve the differential equation and obtain explicit form. For simplicity, we take Λ as above: $\Lambda = K + J$ where K is a complex symmetric matrix.

First we remark the following. For a linear polynomial $l = \sum_{j=1}^{2m} a_j w_j$, we see directly that an ordinary exponential function e^l satisfies

$$e^l \notin \mathcal{E}_1, \quad \in \mathcal{E}_{1+\epsilon}, \quad \forall \epsilon > 0.$$

Then put a Fréchet space

$$\mathcal{E}_{p+} = \cap_{q>p}\mathcal{E}_q.$$

Linear case

P r o p o s i t i o n 3.2.1. *For* $l = \sum_j a_j w_j = <\boldsymbol{a}, \boldsymbol{w}>$, $a_j \in \mathbb{C}$, *we have*

$$\exp_* t\frac{l}{i\hbar} = \exp t^2 \frac{\boldsymbol{a}K\boldsymbol{a}}{4i\hbar} \cdot \exp t\frac{l}{i\hbar} \in \mathcal{E}_{1+}.$$

Quadratic case

P r o p o s i t i o n 3.2.2. *For* $Q_* = \langle \boldsymbol{w}A, \boldsymbol{w}\rangle_*$ *where* A *is a* $2m \times 2m$ *complex symmetric matrix,*

$$\exp_* t(Q_*/i\hbar) = \frac{2^m}{\sqrt{\det M}}\ \exp\frac{1}{i\hbar}\left\langle \boldsymbol{w}\,\frac{J - e^{-2t\alpha}J}{M}\,\boldsymbol{w}\right\rangle,$$

where $M = I - KJ + e^{-2t\alpha}(I + KJ)$ *and* $\alpha = AJ$.

R e m a r k 3.2.1. The star exponentials of linear functions are belonging to $\mathcal{E}_{1+}$ then the star products are convergent and continuous. But it is easy to see

$$\exp_* t(Q_*/i\hbar) \in \mathcal{E}_{2+}, \quad \notin \mathcal{E}_2$$

and hence star exponentials $\exp_* t(Q_*/i\hbar)$ are difficult to treat. Some anomalous phenomena happen, cf. [5].

4. Star functions

There are many applications of star exponential functions, cf. [8]. In this note we show examples using a linear star exponentials.

In what follows, we consider the star product for the simple case where

$$\Lambda = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho \in \mathbb{C}.$$

Then we see easily that the star product is commutative and explicitly given by

$$p_1 *_{\Lambda} p_2 = p_1 \exp\left(\frac{i\hbar\rho}{2} \overleftarrow{\partial_{w_1}}\, \overrightarrow{\partial_{w_1}}\right) p_2.$$

This means that the algebra is essentially reduced to the space of functions of one variable w_1. Thus, we consider functions $f(w)$, $g(w)$ of one variable $w \in \mathbb{C}$ and we consider a commutative star product $*_{\tau}$ with complex parameter τ such that

$$f(w) *_{\tau} g(w) = f(w) \exp\left\{\frac{\tau}{2} \overleftarrow{\partial}_w \overrightarrow{\partial}_w\right\} g(w).$$

4.1. Star Hermite function

Recall the identity

$$\exp\left(\sqrt{2}tw - \frac{1}{2}t^2\right) = \sum_{n=0}^{\infty} H_n(w)\frac{t^n}{n!},$$

where $H_n(w)$ is an Hermite polynomial. We remark here that

$$\exp\left(\sqrt{2}tw - \frac{1}{2}t^2\right) = \exp_*(\sqrt{2}tw_*)_{\tau=-1}.$$

Since $\exp_*(\sqrt{2}tw_*) = \sum_{n=0}^{\infty}(\sqrt{2}tw_*)^n \dfrac{t^n}{n!}$ we have

$$H_n(w) = (\sqrt{2}tw_*)^n\Big|_{\tau=-1}.$$

We define $*$-Hermite function by

$$H_n(w, \tau) = (\sqrt{2}tw_*)^n, \quad (n = 0, 1, 2, \cdots),$$

with respect to $*_\tau$ product. Then we have

$$\exp_*(\sqrt{2}tw_*) = \sum_{n=0}^{\infty} H_n(w,\tau)\,\frac{t^n}{n!}.$$

Trivial identity $\dfrac{d}{dt}\exp_*(\sqrt{2}tw_*) = \sqrt{2}w*\exp_*(\sqrt{2}tw_*)$ yields at every $\tau\in\mathbb{C}$ the identity

$$\frac{\tau}{\sqrt{2}}H_n'(w,\tau) + \sqrt{2}wH_n(w,\tau) = H_{n+1}(w,\tau), \quad (n=0,1,2,\cdots).$$

The exponential law $\exp_*(\sqrt{2}sw_*)*\exp_*(\sqrt{2}tw_*) = \exp_*(\sqrt{2}(s+t)w_*)$ yields at every $\tau\in\mathbb{C}$ the identity

$$\sum_{k+l=n}\frac{n!}{k!\,l!}H_k(w,\tau)*_\tau H_l(w,\tau) = H_n(w,\tau).$$

4.2. Star theta function

In this note we consider the Jacobi's theta functions by using star exponentials as an application.

A direct calculation gives

$$\exp_{*_\tau} i\; tw = \exp(i\; tw - (\tau/4)t^2).$$

Hence for $\mathrm{Re}\,\tau > 0$, the star exponential $\exp_{*_\tau} ni\; w = \exp(ni\; w - (\tau/4)n^2)$ is rapidly decreasing with respect to integer n and then we can consider summations for τ satisfying $\mathrm{Re}\,\tau > 0$

$$\sum_{n=-\infty}^{\infty}\exp_{*_\tau} 2ni\; w = \sum_{n=-\infty}^{\infty}\exp\left(2ni\; w - \tau\, n^2\right) = \sum_{n=-\infty}^{\infty} q^{n^2}e^{2ni\; w}, \quad (q=e^{-\tau}).$$

This is Jacobi's theta function $\theta_3(w,\tau)$. Then we have expression of theta functions as

$$\theta_{1*_\tau}(w) = \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n\exp_{*_\tau}(2n+1)i\; w,$$

$$\theta_{2*_\tau}(w) = \sum_{n=-\infty}^{\infty}\exp_{*_\tau}(2n+1)i\; w,$$

$$\theta_{3*_\tau}(w) = \sum_{n=-\infty}^{\infty}\exp_{*_\tau} 2ni\; w,$$

$$\theta_{4*_\tau}(w) = \sum_{n=-\infty}^{\infty}(-1)^n\exp_{*_\tau} 2ni\; w.$$

Remark that $\theta_{k*_\tau}(w)$ is the Jacobi's theta function $\theta_k(w,\tau)$, $k=1,2,3,4$, respectively. It is obvious by the exponential law

$$\exp_{*_\tau} 2i\ w *_\tau \theta_{k*_\tau}(w) = \theta_{k*_\tau}(w) \quad (k=2,3),$$
$$\exp_{*_\tau} 2i\ w *_\tau \theta_{k*_\tau}(w) = -\theta_{k*_\tau}(w) \quad (k=1,4).$$

Then using $\exp_{*_\tau} 2i\ w = e^{-\tau}e^{2i\ w}$ and the product formula directly we have

$$e^{2i\ w-\tau}\theta_{k*_\tau}(w+i\tau) = \theta_{k*_\tau}(w) \quad (k=2,3),$$
$$e^{2i\ w-\tau}\theta_{k*_\tau}(w+i\tau) = -\theta_{k*_\tau}(w) \quad (k=1,4).$$

4.3. $*$-delta functions

Since the $*_\tau$-exponential $\exp_*(itw_*) = \exp(itw - (\tau/4)t^2)$ is rapidly decreasing with respect to t when $\operatorname{Re}\tau > 0$, then the integral of $*_\tau$-exponential

$$\int_{-\infty}^{\infty} \exp_*(it(w-a)_*)\,dt = \int_{-\infty}^{\infty} \exp_*(it(w-a)_*)dt = \int_{-\infty}^{\infty} \exp(it(w-a) - (\tau/4)t^2)dt$$

converges for any $a \in \mathbb{C}$. We put a star δ-function

$$\delta_*(w-a) = \int_{-\infty}^{\infty} \exp_*(it(w-a)_*)dt,$$

which has a meaning at τ with $\operatorname{Re}\tau > 0$. It is easy to see that for any element $p_*(w) \in \mathcal{P}_*(\mathbb{C})$, we have

$$p_*(w) * \delta_*(w-a) = p(a)\delta_*(w-a),\ w_* * \delta_*(w) = 0.$$

Using the Fourier transform we have

P r o p o s i t i o n 4.3.1.

$$\theta_{1*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty} (-1)^n \delta_*(w + \tfrac{\pi}{2} + n\pi)$$
$$\theta_{2*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty} (-1)^n \delta_*(w + n\pi)$$
$$\theta_{3*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta_*(w + n\pi)$$
$$\theta_{4*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta_*(w + \tfrac{\pi}{2} + n\pi).$$

Now, we consider the τ with the condition $\operatorname{Re}\tau > 0$. Then we calculate the integral and obtain

$$\delta_*(w-a) = \frac{2\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}(w-a)^2\right).$$

Then we have

$$\begin{aligned}\theta_3(w,\tau) &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\delta_*(w+n\pi) = \sum_{n=-\infty}^{\infty}\frac{\sqrt{\pi}}{\sqrt{\tau}}\exp\left(-\frac{1}{\tau}(w+n\pi)^2\right)\\ &= \frac{\sqrt{\pi}}{\sqrt{\tau}}\exp\left(-\frac{1}{\tau}\right)\sum_{n=-\infty}^{\infty}\exp\left(-2n\frac{1}{\tau}w-\frac{1}{\tau}n^2\tau^2\right)\\ &= \frac{\sqrt{\pi}}{\sqrt{\tau}}\exp\left(-\frac{1}{\tau}\right)\theta_{3*}(\frac{2\pi w}{i\tau},\frac{\pi^2}{\tau}).\end{aligned}$$

We also have similar identities for other $*$-theta functions by the similar way.

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