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ON DISCRETENESS OF SPECTRUM OF A SECOND ORDER FUNCTIONAL DIFFERENTIAL OPERATOR

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Abstract. Necessary and sufficient conditions for discreteness of spectrum for the singular second order functional differential operator of the form

$$\frac{1}{\rho(x)}\left(-(p(x)u')' + q(x)u - \int_a^b u(s)r(x,ds)\right),\ x \in (a,b),\ -\infty \le a < b \le \infty$$

are obtained.

Keywords: discreteness of spectrum; functional differential operator

1. The main results.

Let $\mathcal{L}$ be the functional differential operator defined by

$$\mathcal{L}u(x) := \frac{1}{\rho(x)}\left((-p(x)u')' + q(x)u - \int_a^b u(s)r(x,ds)\right),\ x \in (a,b), \tag{1.1}$$

$-\infty \le a < b \le \infty$. Let $I := (a,b)$. Recall that the spectrum of an operator A acting in a Hilbert space H is discrete if it consists only of eigenvalues of finite multiplicity (see, for example, [7]). Let $L_2(I,\rho)$ be the space of square integrable on I with a positive measurable weight ρ functions. In this space the question about discreteness of spectrum of the differential operator

$$\mathcal{L}u = \frac{1}{\rho}\left(-(pu')' + qu\right),\ x \in I = (a,b), \tag{1.2}$$

is well studied. In the case $(a,b) = (-\infty,\infty)$, for the operator $-u'' + qu$ a simple sufficient condition $\lim_{x\to\infty} q(x) = +\infty$ was obtained by K. Friedrichs [3]. The following necessary and sufficient condition was obtained by A. M. Molchanov [1]:

$$(\forall\delta > 0)\ \lim_{x\to\infty}\int_x^{x+\delta} q(x)dx = +\infty. \tag{1.3}$$

Note that Molchanov studied the n-dimensional space R^n. The condition (1.3) is a special particular case for $n = 1$. In the case $q = 0$ for the operator $-(1/\rho)(pu')'$ a necessary and sufficient condition is obtained by I. Kac and M.G. Krein [2]. M. Birman [6] obtained a necessary and sufficient condition for an operator of even order on semiaxis $[0,\infty)$. Under condition $\int_0^\infty \rho(x)\,dx < \infty$ for the operator

$$\mathcal{L}_0 u = -(1/\rho)u''$$

the Birman's condition has the form

$$\lim_{s\to\infty} s\int_s^\infty \rho(x)\,dx = 0. \tag{1.4}$$

If $\int_0^1 \rho(x)\,dx = \infty$, the condition

$$\lim_{s\to 0} s\int_s^\infty \rho(x)\,dx = 0$$

together with (1.4) guaranties discreteness of spectrum of $-(1/\rho)u''$. This second singularity is by cause of non-integrability of ρ at the point $x=0$. This follows from the result of Kac and Krein [2] but it is not easy to see this at once.

Assume that the functions ρ, p and q are measurable, ρ, p are positive in I, the function q is non-negative. For almost all $x\in I$ the $r(x,\cdot)$ is a measure, that can be defined by a non-decreasing function $r(x,s)$. Assume that

$$q(x) \ge r(x,I)$$

almost everywhere on I. Assume that $1/p$ and ρ are locally integrable in I, that is

$$\int_{s_1}^{s_2} \frac{dx}{p(x)} < \infty,\ \int_{s_1}^{s_2} \rho(x)\,dx < \infty,\ (a < s_1 < s_2 < b). \tag{1.5}$$

Say that $\mathcal{L}$ has singularity at $x=a$ by $p(x)$ if

$$\int_a^s \frac{dx}{p(x)} = \infty,\ a < s < b.$$

Analogously we mean singularities by ρ and at $x=b$ (4 cases). It is clear that the singularity at the right end of the interval can be considered similarly to the left. Moreover, the singularity at the right end can be reduced to the singularity at the left end by the change of variable $x=-x'$. Thus, we can consider some singularity only at the left end of I.

$$\int_s^b \frac{dx}{p(x)} < \infty \text{ and } \int_s^b \rho(x)\,dx < \infty \quad (a < s < b). \tag{1.6}$$

Only one type of singularity is allowed. We have to consider two cases: the first is

$$\int_a^s \rho(x)\,dx = \infty,\ \int_a^s \frac{dx}{p(x)} < \infty \tag{1.7}$$

and the second is

$$\int_a^s \frac{dx}{p(x)} = \infty,\ \int_a^s \rho(x)\,dx < \infty \tag{1.8}$$

for any $s\in I$. Let

$$\Phi_1(s) = \int_a^s \frac{dx}{p(x)} \int_s^l \rho(x)\,dx, \quad \Phi_2(s) = \int_a^s \rho(x)\,dx \int_s^l \frac{dx}{p(x)}$$

Theorem 1. *Suppose one of the following conditions holds:*

$$\lim_{s\to 0} \Phi_1(s) = 0 \ \textit{or} \ \lim_{s\to 0} \Phi_2(s) = 0.$$

Then the spectrum of the operator $\mathcal{L}$ *is discrete. These conditions are necessary if for example the function* $q(x)$ *is bounded.*

R e m a r k 1. From this theorem it follows the sufficient discreteness condition in [4,5].

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О ДИСКРЕТНОСТИ СПЕКТРА ФУНКЦИОНАЛЬНО-ДИФФЕРЕНЦИАЛЬНОГО ОПЕРАТОРА ВТОРОГО ПОРЯДКА

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Аннотация. Получены необходимые и достаточные условия дискретности спектра для сингулярного дифференциального оператора вида

$$\frac{1}{\rho(x)}\left(-(p(x)u')' + \int_a^b u(s)r(x,ds)\right),\ x \in (a,b),\ -\infty \le a < b \le \infty.$$

Ключевые слова: дискретность спектра; функционально-дифференциальный оператор

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