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Hermite functions and inner product in Sobolev space

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Abstract. Let us consider the orthogonal Hermite system $\{\varphi_{2n}(x)\}_{n\geq 0}$ of even index defined on $(-\infty,\infty)$, where

$$\varphi_{2n}(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{(2n)!}\pi^{\frac{1}{4}}2^n}H_{2n}(x),$$

by $H_{2n}(x)$ we denote a Hermite polynomial of degree $2n$. In this paper, we consider a generalized system $\{\psi_{r,2n}(x)\}$ with $r>0$, $n\geq 0$ which is orthogonal with respect to the Sobolev type inner product on $(-\infty,\infty)$, i.e.

$$\langle f,g\rangle=\lim_{t\to-\infty}\sum_{k=0}^{r-1}f^{(k)}(t)g^{(k)}(t)+\int_{-\infty}^{\infty}f^{(r)}(x)g^{(r)}(x)\rho(x)dx$$

with $\rho(x)=e^{-x^2}$, and generated by $\{\varphi_{2n}(x)\}_{n\geq 0}$. The main goal of this work is to study some properties related to the system $\{\psi_{r,2n}(x)\}_{n\geq 0}$,

$$\psi_{r,n}(x)=\frac{(x-a)^n}{n!},\quad n=0,1,2,\ldots,r-1,$$
$$\psi_{r,r+n}(x)=\frac{1}{(r-1)!}\int_a^b(x-t)^{r-1}\varphi_n(t)dt,\quad n=0,1,2,\ldots.$$

We study the conditions on a function $f(x)$, given in a generalized Hermite orthogonal system, for it to be expandable into a generalized mixed Fourier series as well as the convergence of this Fourier series. The second result of the paper is the proof of a recurrent formula for the system $\{\psi_{r,2n}(x)\}_{n\geq 0}$. We also discuss the asymptotic properties of these functions, and this concludes our contribution.

Keywords: inner product, Sobolev space, Hermite polynomials

Mathematics Subject Classification: 42C10.

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НАУЧНАЯ СТАТЬЯ

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Функции Эрмита и скалярное произведение в пространстве Соболева

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Аннотация. Рассмотрим ортогональную систему Эрмита $\{\varphi_{2n}(x)\}_{n\geq 0}$ четного индекса, определенную на $(-\infty,\infty)$ формулой

$$\varphi_{2n}(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{(2n)!}\pi^{\frac{1}{4}}2^n}H_{2n}(x),$$

где через $H_{2n}(x)$ обозначен полином Эрмита степени $2n$. В данной работе рассматривается обобщенная система $\{\psi_{r,2n}(x)\}$ с $r>0$, $n\geq 0$, ортогональная относительно скалярного произведения Соболевского типа на $(-\infty,\infty)$

$$\langle f,g\rangle=\lim_{t\to-\infty}\sum_{k=0}^{r-1}f^{(k)}(t)g^{(k)}(t)+\int_{-\infty}^{\infty}f^{(r)}(x)g^{(r)}(x)\rho(x)dx$$

с $\rho(x)=e^{-x^2}$, и порожденная системой $\{\varphi_{2n}(x)\}_{n\geq 0}$. Основной целью работы является изучение некоторых свойств, связанных с системой $\{\psi_{r,2n}(x)\}_{n\geq 0}$,

$$\psi_{r,n}(x)=\frac{(x-a)^n}{n!},\quad n=0,1,2,\dots,r-1,$$
$$\psi_{r,r+n}(x)=\frac{1}{(r-1)!}\int_a^b(x-t)^{r-1}\varphi_n(t)dt,\quad n=0,1,2,\dots .$$

Изучаются условия на функцию $f(x)$, заданную в обобщенной ортогональной системе Эрмита, достаточные для ее разложения в обобщенный смешанный ряд Фурье, а также сходимость этого ряда Фурье. Второй результат статьи — доказательство рекуррентной формулы для системы $\{\psi_{r,2n}(x)\}_{n\geq 0}$. Также обсуждаются асимптотические свойства этих функций, что составляет заключительную часть работы.

Ключевые слова: скалярное произведение, пространство Соболева, многочлены Эрмита

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Introduction

Consider an orthogonal system $\{\varphi_n(x)\}_{n\geq 0}$ on $(-\infty,\infty)$ with $\rho(x)$ as a weight function, and let $r>0$. We construct a new orthogonal system $\{\psi_{r,n}(x)\}_{n\geq 0}$ by following the Sobolev type inner product:

$$\langle f,g\rangle_S=\lim_{t\to-\infty}\sum_{v=0}^{r-1} f^{(v)}(t)g^{(v)}(t)+\int_{-\infty}^{\infty} f^{(r)}(x)g^{(r)}(x)\rho(x)dx. \tag{0.1}$$

Quite a few authors have presented this type of construction. For example, I. I. Sharapudinov, in his works [1–4] on the construction of mixed Fourier series, considered some particular cases of systems generated by classes of orthogonal functions, namely, those of Jacobi, Legendre, Chebychev, Laguerre, and Haar. The case of the Gegenbauer system is covered in [5].

Hermite polynomials are widely used in mechanics and physics, and are of particular interest for applications. In this work, we reconstruct the orthogonal system $\{\psi_{r,n}(x)\}_{n\geq 0}$ generated by Hermite polynomials of even index, and this approach is different from that used by M. A. Boudref.

Denote by $L^p_\rho(a,b)$ the space of measurable functions $f(x)$, $x\in(a,b)$, such that

$$\int_a^b |f(x)|^p\rho(x)dx<\infty.$$

It is clear that $L^p_\rho(a,b)$ is a Banach space with the norm

$$\|f\|_{p,\rho}=\left(\int_a^b |f(x)|^p\rho(x)dx\right)^{\frac{1}{p}}.$$

When $\rho(x)=1$, we write $L^p_\rho(a,b)=L^p(a,b)$.

We can define functions of the system $\{\psi_{r,n}(x)\}_{n\geq 0}$ for $r>0$ as follows [4]:

$$\begin{aligned}\psi_{r,n}(x)&=\frac{(x-a)^n}{n!},\quad n=0,1,2,\ldots,r-1,\\ \psi_{r,r+n}(x)&=\frac{1}{(r-1)!}\int_a^b (x-t)^{r-1}\varphi_n(t)dt,\quad n=0,1,2,\ldots.\end{aligned} \tag{0.2}$$

From (0.2), we have for a. e. $x\in(a,b)$

$$\psi^{(v)}_{r,n}(x)=\begin{cases}\psi_{r-v,n-v}(x), & \text{if } 0\leq v\leq r-1,\ r\leq n,\\ \varphi_{n-v}(x), & \text{if } v=r\leq n,\\ \psi_{r-v,n-v}(x), & \text{if } v\leq n<r,\\ 0, & \text{if } n<v\leq r.\end{cases}$$

Denote by $W^r_{L^p_\rho(a,b)}$ the Sobolev weighted space. This space consists of all functions $f(x)$ which are $r-1$ times continuously differentiable on the closed interval $[a,b]$ and such that $f^{(r-1)}(x)$ is absolutely continuous on $[a,b]$ and $f^{(r)}(x)\in L^p_\rho(a,b)$.

For $p=2$, we define in $W^r_{L^2_\rho(a,b)}$ the inner product by (0.1). For any function $f\in W^r_{L^2_\rho(a,b)}$, we can set the norm by

$$\|f\|_{W^r_{L^2_\rho(a,b)}}=\sqrt{\langle f,f\rangle_S},$$

which allows us to deduce that $\left(W^r_{L^2_\rho(a,b)},\|\cdot\|_{W^r_{L^2_\rho(a,b)}}\right)$ is a Banach space, and $\left\langle W^r_{L^2_\rho(a,b)},\langle\cdot,\cdot\rangle_S\right\rangle$ is a Hilbert space.

The system $\{\psi_{r,n}(x)\}_{n\geq 0}$ is said to be Sobolev-orthogonal with respect to the inner product (0.1) generated by the system $\{\varphi_n(x)\}_{n\geq 0}$.

1. Some properties of Hermite polynomials

Let $\psi_{r,n}(x)$ be Sobolev-orthogonal polynomials with respect to the inner product

$$\langle f,g\rangle=\lim_{t\to-\infty}\sum_{v=0}^{r-1}f^{(v)}(t)g^{(v)}(t)+\int_{-\infty}^{\infty}f^{(r)}(x)g^{(r)}(x)w(x)dx,$$

where $w(x)=e^{-x^2}$.

Here are some properties of Hermite polynomials:

- The Hermite polynomials are given by [6]

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right),\quad x\in\mathbb{R},$$

where

$$H_0(x)=1,\quad H_1(x)=2x,\quad H_2(x)=4x^2-2.$$

- Recurrence formula [7, p. 106]:

$$H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x),\quad H'_{n+1}(x)=2(n+1)H_n(x),$$
$$H_n(x)=2xH_{n-1}(x)-H'_{n-1}(x).$$

- Orthogonality formula:

$$\int_{-\infty}^{\infty}e^{-x^2}H_n(x)H_m(x)dx=\delta_{nm}c_n,\quad\text{where}\quad c_n=2^n n!\sqrt{\pi}.$$

A Hermite function is defined by

$$\varphi_n(x)=e^{-\frac{x^2}{2}}H_n(x).$$

- We define a Hermite function of even index by

$$\varphi_{2n}(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{(2n)!}\pi^{\frac{1}{4}}2^n}H_{2n}(x),$$

where $H_{2n}(x)$ is a Hermite polynomial of even index, in particular,

$$H_{2n}(0)=(-1)^n\frac{(2n)!}{n!},\quad H_{2n+1}(0)=(-1)^n\frac{(2n+2)!}{(n+1)!}.$$

- Relation to Legendre functions:

$$H_{2n}(x)=(-1)^n 2^{2n}L_n^{-\frac{1}{2}}\left(x^2\right).$$

So if

$$L_n^{-\frac{1}{2}}\left(x^2\right)=x^{-1}e^{x^2}\frac{d^n}{dx^n}\left(x^{2\left(-\frac{1}{2}+n\right)}e^{-x^2}\right),$$

then

$$H_{2n}(x)=(-1)^n 2^{2n}x^{-1}e^{x^2}\frac{d^n}{dx^n}\left(x^{2\left(-\frac{1}{2}+n\right)}e^{-x^2}\right).$$

- Asymptotic formula for $\varphi_{2n}(x)$ [8, p. 594]:

$$\varphi_{2n}(x)=\varphi_{2n}(0)\left[\cos\left(\sqrt{4n+1}x\right)+\frac{x^{\frac{5}{2}}}{\sqrt[4]{4n+1}}\theta_n(x)\right],\quad \text{where}\quad -1<\theta_n(x)<1.$$

It is clear that

$$\lim_{n\to\infty}\left[\cos\left(\sqrt{4n+1}x\right)+\frac{x^{\frac{5}{2}}}{\sqrt[4]{4n+1}}\theta_n(x)\right]=0,\quad \text{for each } x.$$

So, the asymptotic formula for $H_{2n}(x)$ is

$$H_{2n}(x)=(-1)^n2^n\cdot 1\cdot 3\cdot 5\cdot \ldots\cdot(2n-1)e^{\frac{1}{2}x^2}\left[\cos\left(\sqrt{4n+1}x\right)+o\left(\frac{1}{\sqrt[4]{n}}\right)\right].$$

- The system $\{\varphi_{2n}(x)\}_{n\geq 1}$ is orthogonal on $(-\infty,\infty)$, i. e.

$$\int_{-\infty}^{\infty}\varphi_{2n}(x)\varphi_{2m}(x)dx=\delta_{nm}.$$

- Some particular cases:

$$\varphi_0(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt[4]{\pi}}H_0(x),\quad \varphi_2(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{2}\sqrt[4]{\pi}}H_2(x).$$

1.1. Sobolev-orthogonal functions generated by Hermite functions $\varphi_{2n}(x)$

D e f i n i t i o n 1.1. For $r>0$, we define the functions $\psi_{r,2n}(x)$ $(n=0,1,\ldots)$ by

$$\psi_{r,2n}(x)=\frac{(x)^{2n}}{(2n)!},\quad n=0,1,\ldots,r-1,$$
$$\psi_{r,r+2n}(x)=\frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1}\varphi_{2n}(t)dt,\quad n\geq 0.$$

We will calculate the functions $\psi_{r,r+2n}(x)$ for any $n\in\mathbb{N}$ and $x\in(-\infty,\infty)$.

Theorem 1.1 (Fisrt aim result)**.** *For $n\geq 1$ and $r>0$, we have the following relations:*

1. $\psi_{r,r+2n}(x)=-r\sqrt{\frac{2}{2n-1}}\psi_{r+1,r+2n}(x)+x\sqrt{\frac{2}{2n-1}}\psi_{r,r+2n-1}(x)-\sqrt{\frac{2n-1}{2n}}\psi_{r,r+2n-2}(x).$

2. $\psi_{r+1,r+1+2n}(x)=\frac{x}{r}\psi_{r,r+2n}(x)+\frac{1}{r}\psi_{r-1,r+2n-1}(x)+\frac{2\sqrt{n}}{r}\psi_{r,r+2n-1}(x).$

3. *For $n=0$:*

$$\psi_{r,r}(x)=\frac{-\sqrt{2}}{1-\sqrt{2}}\psi_{r+1,r+2}(x)+\frac{x\sqrt{2}}{1-\sqrt{2}}\psi_{r,r+1}(x).$$

To prove this theorem, we need the following lemma.

Lemma 1.1. *The formulas for the derivations of the Hermite functions of even index are:*

$$\varphi_{2n}(x)=\frac{\sqrt{2}}{\sqrt{2n-1}}x\varphi_{2n-1}(x)-\sqrt{1-\frac{1}{2n}}\varphi_{2n-2}(x), \tag{1.1}$$

$$\varphi'_{2n}(x)=-x\varphi_{2n}(x)+2\sqrt{n}\varphi_{2n-1}(x). \tag{1.2}$$

P r o o f. First, for the Hermite functions, we put for $n\geq 1$

$$\varphi_{2n}(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{(2n)!}\pi^{\frac14}2^n}H_{2n}(x),\quad \varphi_{2n-1}(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{(2n-1)!}\pi^{\frac14}2^{n-\frac12}}H_{2n-1}(x),$$

$$\varphi_{2n-2}(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{(2n-2)!}\pi^{\frac14}2^{n-1}}H_{2n-2}(x).$$

Use these formulas to prove (1.1), (1.2).

a) We have (see [7, p. 106])

$$H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x).$$

Multiplying both sides by $\dfrac{e^{-\frac{x^2}{2}}}{\sqrt{(2n)!}\pi^{\frac14}2^n}$, we get

$$\begin{aligned}\varphi_{2n}(x) &= \frac{2xe^{-\frac{x^2}{2}}}{\sqrt{(2n)!}\pi^{\frac14}2^n}H_{2n-1}(x)-2(2n-1)\frac{e^{-\frac{x^2}{2}}}{\sqrt{(2n)!}\pi^{\frac14}2^n}H_{2n-2}(x)\\ &= \sqrt{\frac{2}{2n-1}}x\frac{e^{-\frac{x^2}{2}}}{\sqrt{(2n-1)!}\pi^{\frac14}2^{n-\frac12}}H_{2n-1}(x)-\sqrt{\frac{2n-1}{2n}}\frac{e^{-\frac{x^2}{2}}}{\sqrt{(2n-2)!}\pi^{\frac14}2^{n-1}}H_{2n-2}(x),\end{aligned}$$

and so we have (1.1).

b) For the second relation, we differentiate

$$\varphi_{2n}(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{(2n)!}\pi^{\frac14}2^n}H_{2n}(x)$$

with respect to x, and get

$$\varphi'_{2n}(x)=-x\frac{e^{-\frac{x^2}{2}}}{\sqrt{(2n)!}\pi^{\frac14}2^n}H_{2n}(x)+\frac{e^{-\frac{x^2}{2}}}{\sqrt{(2n)!}\pi^{\frac14}2^n}H'_{2n}(x).$$

Using the fact that $H'_{2n}(x)=4nH_{2n-1}(x)$, we get

$$\begin{aligned}\varphi'_{2n}(x) &= -x\frac{e^{-\frac{x^2}{2}}}{\sqrt{(2n)!}\pi^{\frac14}2^n}H_{2n}(x)+\frac{e^{-\frac{x^2}{2}}}{\sqrt{(2n)!}\pi^{\frac14}2^n}H'_{2n}(x)\\ &= -x\varphi_{2n}(x)+\frac{2^{\frac32}ne^{-\frac{x^2}{2}}}{\sqrt{n}\sqrt{(2n-1)!}\pi^{\frac14}2^{n-\frac12}}H_{2n-1}(x),\end{aligned}$$

so $\varphi'_{2n}(x)=-x\varphi_{2n}(x)+2\sqrt{n}\varphi_{2n-1}(x)$. □

P r o o f. (of Theorem 1.1)

1. First of all, it is clear that

$$\psi_{0,2n}(x)=\varphi_{2n}(x),\quad \psi_{1,0}(x)=1,\quad \psi_{1,1}(x)=\int_{-\infty}^{x}\varphi_0(t)dt.$$

We have

$$\psi_{r,r+2n}(x)=\frac{1}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\varphi_{2n}(t)dt,\quad \text{where}\quad \varphi_{2n}(t)=\frac{e^{-\frac{t^2}{2}}}{\sqrt{(2n)!\pi^{\frac{1}{4}}2^n}}H_{2n}(t). \tag{1.3}$$

Then, from Lemma 1.1 and (1.3), it follows that

$$\begin{aligned}\psi_{r,r+2n}(x)=&\frac{1}{(r-1)!}\int_{-\infty}^{x}\sqrt{\frac{2}{2n-1}}t(x-t)^{r-1}\varphi_{2n}(t)dt-\frac{\sqrt{\frac{2n-1}{2n}}}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\varphi_{2n-2}(t)dt\\ =&\frac{\sqrt{\frac{2}{2n-1}}}{(r-1)!}\int_{-\infty}^{x}t\,(x-t)^{r-1}\,\varphi_{2n-1}(t)dt-\frac{\sqrt{\frac{2}{2n-1}}}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\,\varphi_{2n-2}(t)dt.\end{aligned}$$

In the second term of this expression, we have

$$\psi_{r,r+n-2}(x)=\frac{1}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\varphi_{2n-2}(t)dt.$$

Let us calculate

$$J\doteq\frac{1}{(r-1)!}\int_{-\infty}^{x}t(x-t)^{r-1}\varphi_{2n-1}(t)dt.$$

We get

$$\begin{aligned}J&=\frac{1}{(r-1)!}\int_{-\infty}^{x}(t-x+x)(x-t)^{r-1}\varphi_{2n-1}(t)dt\\ &=-\frac{1}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r}\varphi_{2n-1}(t)dt+\frac{x}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\varphi_{2n-1}(t)dt\\ &=-r\psi_{r+1,r+2n}(x)+x\psi_{r,r+2n-1}(x).\end{aligned}$$

Then

$$\psi_{r,r+2n}(x)=-r\sqrt{\frac{2}{2n-1}}\psi_{r+1,r+2n}(x)+x\sqrt{\frac{2}{2n-1}}\psi_{r,r+2n-1}(x)-\sqrt{\frac{2n-1}{2n}}\psi_{r,r+2n-2}(x).$$

2. By Lemma 1.1 (formula (1.2)), we have

$$-x\varphi_{2n}(t)=\varphi_{2n}^{'}(x)-2\sqrt{n}\varphi_{2n-1}(x),$$

so

$$\begin{aligned}-\frac{1}{(r-1)!}\int_{-\infty}^{x}t(x-t)^{r-1}\varphi_{2n}(t)dt=&\frac{1}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\varphi_{2n}^{'}(t)dt\\ &-\frac{2\sqrt{n}}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\varphi_{2n-1}(t)dt.\end{aligned}$$

Let us calculate

$$H_1 \doteq \frac{1}{(r-1)!}\int_{-\infty}^{x} t(x-t)^{r-1}\varphi_{2n}(t)dt, \quad H_2 \doteq \frac{1}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\varphi_{2n}'(t)dt,$$

$$H_3 \doteq \frac{1}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\varphi_{2n-1}(t)dt.$$

First,

$$H_1 = \frac{1}{(r-1)!}\int_{-\infty}^{x} t(x-t)^{r-1}\varphi_{2n}(t)dt = \frac{1}{(r-1)!}\int_{-1}^{x}(t-x+x)\,(x-t)^{r-1}\,\varphi_{2n}(t)dt$$
$$= -\frac{r}{r!}\int_{-1}^{x}(x-t)^{r}\varphi_{2n}(t)dt + \frac{x}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1}\varphi_{2n}(t)dt = -r\psi_{r+1,r+2n}(x) + x\psi_{r,r+2n}(x).$$

For H_2, integrating by parts we get

$$H_2 = \frac{1}{(r-1)!}\left[(x-t)^{r-1}\,\varphi_{2n}(t)\right]_{-\infty}^{x} + \frac{1}{(r-2)!}\int_{-\infty}^{x}(x-t)^{r-2}\,\varphi_{2n}(t)dt.$$

Since

$$\lim_{t\to-\infty}(x-t)^{r-1}\,\varphi_{2n}(t) = \lim_{t\to-\infty}(x-t)^{r-1}\,\frac{e^{-\frac{t^2}{2}}}{\sqrt{(2n)!}\pi^{\frac{1}{4}}2^{n}} = 0,$$

then

$$H_2 = \frac{1}{(r-2)!}\int_{-\infty}^{x}(x-t)^{r-2}\,\varphi_{2n}(t)dt = \psi_{r-1,r+2n-1}(x).$$

Regarding H_3, we have

$$H_3 = \frac{1}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\varphi_{2n-1}(t)dt = \psi_{r,r+2n-1}(x).$$

Finally,

$$\psi_{r,r+1+2n}(x) = \frac{x}{r}\psi_{r,r+2n}(x) + \frac{1}{r}\psi_{r-1,r+2n-1}(x) + \frac{2\sqrt{n}}{r}\psi_{r,r+2n-1}(x).$$

So we obtain the second formula.

3. For this part, it is easy to see that for $n=0$, $\varphi_0(x) = x\sqrt{2}\varphi_1(x) + \sqrt{2}\varphi_2(x)$. Then

$$\psi_{r,r}(x) = \frac{1}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\varphi_0(t)dt$$
$$= \frac{\sqrt{2}}{(r-1)!}\int_{-\infty}^{x} t(x-t)^{r-1}\varphi_1(t)dt + \frac{\sqrt{2}}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\varphi_0(t)dt$$
$$= \frac{-\sqrt{2}}{(r-1)!}\int_{-\infty}^{x}(x-t-x)(x-t)^{r-1}\varphi_1(t)dt + \frac{\sqrt{2}}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\varphi_0(t)dt,$$

so

$$\psi_{r,r}(x) = \frac{-\sqrt{2}}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r}\varphi_1(t)dt + \frac{\sqrt{2}x}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\varphi_1(t)dt$$
$$+\frac{\sqrt{2}}{(r-1)!}\int_{-\infty}^{x}(x-t)^{r-1}\varphi_0(t)dt,$$
$$\psi_{r,r}(x) = -\sqrt{2}\psi_{r+1,r+2}(x) + x\sqrt{2}\psi_{r,r+1}(x) + \sqrt{2}\psi_{r,r}(x).$$

Then $\psi_{r,r}(x) = -\dfrac{\sqrt{2}}{1-\sqrt{2}}\psi_{r+1,r+2}(x) + \dfrac{x\sqrt{2}}{1-\sqrt{2}}\psi_{r,r+1}(x).$ □

1.2. Problem of mixed Fourier series

Let $f \in W^r_{L^2_\rho(-\infty,\infty)}$. If this function is given in the generalized Hermite orthogonal system $\{\psi_{r,2n}(x)\}_{n\geq 0}$, then

$$f(x) \sim \sum_{k=0}^{\infty} C_{r,k}\psi_{r,2k}(x). \tag{1.4}$$

This Fourier series will have the form:

$$f(x) \sim \lim_{t\to-\infty} \sum_{k=0}^{r-1} f^{(k)}(t)\frac{x^{2k}}{(2k)!} + \sum_{k=r}^{\infty} C_{r,k}\psi_{r,2k}(x), \tag{1.5}$$

with

$$C_{r,k} = \hat{f}_{r,k} = \int_{-\infty}^{\infty} f^{(k)}(t)\psi_{r,2k}(t)dt = \int_{-\infty}^{\infty} f^{(k)}(t)\varphi_{2k-r}(t)dt$$

called the Fourier coefficient. For $r = 0$, we have

$$f(x) \sim \sum_{k=r}^{\infty} C_{0,k}\psi_{0,2k}(x) \sim \sum_{k=r}^{\infty} \hat{f}_{0,k}\psi_{0,2k}(x) \tag{1.6}$$

with

$$\hat{f}_{0,2k} = \int_{-\infty}^{1} f(t)\varphi_{2k}(t)dt.$$

In this section, we will give the expressions of (1.5) and (1.6) taking into account the expression of $\varphi_{2n}(t)$. Also we will prove the convergence of the series (1.4). The following result is similar to the one given in [5].

Theorem 1.2. *For $n \geq 0$, $r > 0$, the system of functions $\{\psi_{r,2n}(x)\}$ generated by Hermite functions $\varphi_{2n}(x)$ by the formula*

$$\psi_{r,r+2n}(x) = \frac{1}{(r-1)!}\int_{-\infty}^{x} (x-t)^{r-1}\varphi_{2n}(t)dt, \quad n \geq 0,$$

is complete in $W^r_{L^2_\rho(-\infty,\infty)}$ and orthonormal via Sobolev's inner product

$$\langle f, g\rangle = \lim_{t\to-\infty} \sum_{v=0}^{r-1} f^{(v)}(t)g^{(v)}(t) + \int_{-\infty}^{\infty} f^{(r)}(t)g^{(r)}(t)w(t)dt.$$

It follows from the formulas

$$\psi_{r,r+2n}(x) = \frac{1}{(r-1)!}\int_{-\infty}^{x} (x-t)^{r-1}\varphi_{2n}(t)dt, \quad n \geq 0,$$

$$\psi_{r,2n}(x) = \frac{x^{2n}}{(2n)!}, \quad n = 0, 1, \ldots, r-1,$$

that for all $x \in (-\infty, \infty)$,

$$(\psi_{r,2n}(x))^{(v)} = \begin{cases} \psi_{r-v,2n-v}(x), & 0 \leq v \leq r-1,\ r \leq 2n, \\ \varphi_{2n-v}(x), & v = r \leq 2n, \\ \psi_{r-v,2n-v}(x), & v \leq 2n < r, \\ 0, & 2n < v \leq r, \end{cases}$$

with $\psi_{0,2n}(x) = \varphi_{2n}(x)$.

1.3. Study of the convergence of the series (1.5)

Let $f \in W^r_{L^2_\rho(-\infty,\infty)}$, then $f^{(r)} \in L^p$ with

$$f^{(r)}(x) \sim \sum_{k=0}^{\infty} \hat{C}_{r,k}\varphi_k(x), \quad \text{where} \quad \hat{C}_{r,k} = \int_{-\infty}^{\infty} f^{(r)}(t)\varphi_k(t)dt, \quad \text{for all } k \geq 0.$$

Theorem 1.3 (Second aim result)**.** *For* $x \in [A, B]$ $(A < B < \infty)$ *and* $f \in W^r_{L^p}$, *where* $\frac{4}{3} < p < 4$, *the Fourier series*

$$f(x) \sim \sum_{k=0}^{r-1} \lim_{t\to-\infty} f^{(k)}(t)\frac{x^{2k}}{(2k)!} + \sum_{k=r}^{\infty} C_{r,k}\psi_{r,2k}(x)$$

converges uniformly to the function f.

P r o o f. We note the following partial sums:

$$S_{r,n}(f, x) = \sum_{k=0}^{r-1} \lim_{t\to-\infty} f^{(k)}(t)\frac{x^{2k}}{(2k)!} + \sum_{k=r}^{n} C_{r,k}\psi_{r,2k}(x), \quad S^*_{r,n}\left(f^{(r)}, x\right) = \sum_{k=0}^{n} \hat{C}_{r,k}\varphi_{2k}(x).$$

Then

$$\left|f(x) - S_{r,r+n}(f, x)\right| = \left|\frac{1}{(r-1)!}\int_A^x (x-t)^{r-1} f^{(r)}(t)dt - \sum_{k=r}^{r+n} C_{r,k}\psi_{r,2k}(x)\right|$$

with

$$\psi_{r,r+2k}(x) = \frac{1}{(r-1)!}\int_A^x (x-t)^{r-1}\varphi_{2n}(t)dt,$$

so

$$\psi_{r,2k}(x) = \frac{1}{(r-1)!}\int_A^x (x-t)^{r-1}\varphi_{2k-r}(t)dt.$$

Hence,

$$|f(x) - S_{r,r+n}(f,x)| = \left|\frac{1}{(r-1)!}\int_A^x (x-t)^{r-1}f^{(r)}(t)dt - \frac{1}{(r-1)!}\sum_{k=r}^{r+n} C_{r,k}\int_A^x (x-t)^{r-1}\varphi_{2k-r}(t)dt\right|$$
$$= \frac{1}{(r-1)!}\left|\int_A^x (x-t)^{r-1}\left(f^{(r)}(t) - \sum_{k=r}^{r+n} C_{r,k}\varphi_{2k-r}(t)\right)dt\right|$$

with

$$\sum_{k=r}^{r+n} C_{r,k}\varphi_{2k-r}(t) = S^*_{r,n}\left(f^{(r)}, x\right).$$

Then

$$|f(x) - S_{r,r+n}(f, x)| \leq \frac{1}{(r-1)!}\int_A^x (x-t)^{r-1}\left|f^{(r)}(t) - S^*_{r,n}\left(f^{(r)}, t\right)\right| dt.$$

Using Hölder's inequality, we get:

$$|f(x) - S_{r,r+n}(f, x)| \leq \frac{1}{(r-1)!}\left(\int_A^x (x-t)^{(r-1)q}dt\right)^{\frac{1}{q}}\left(\int_A^x \left|f^{(r)}(t) - S^*_{r,n}\left(f^{(r)}, t\right)\right|^p dt\right)^{\frac{1}{p}},$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Calculate

$$\int_A^x (x-t)^{(r-1)q}dt = (-1)^{q(r-1)}\frac{(t-x)^{q(r-1)+1}}{q(r-1)+1}\Big|_A^x$$
$$= (-1)^{q(r-1)}\frac{(A-x)^{q(r-1)+1}}{q(r-1)+1} = \frac{(B-A)^{q(r-1)+1}}{q(r-1)+1} < \infty.$$

Then

$$|f(x) - S_{r,r+n}(f,x)| \leq \frac{1}{(r-1)!}\left(\frac{(B-A)^{q(r-1)+1}}{q(r-1)+1}\right)^{\frac{1}{q}} \left\| f^{(r)}(x) - S^*_{r,n}\left(f^{(r)},x\right)\right\|_{L^p},$$

and since $\left\| f^{(r)}(x) - S^*_{r,n}\left(f^{(r)},x\right)\right\|_{L^p} \to 0$ as $n\to\infty$, it results that $|f(x)-S_{r,r+n}(f,x)|\to 0$, uniformly on $[A,B]$. □

2. Asymptotic forms of the functions $\psi_{1,1+2n}(x)$

We say that

$$\psi_{1,1+2n}(x) = \int_{-\infty}^x \varphi_{2n}(t)dt = \frac{1}{2^n\sqrt{(2n)!}\pi^{\frac14}}\int_{-\infty}^x e^{-\frac{t^2}{2}}H_{2n}(t)dt.$$

Integrating by parts and using the first formula of Lemma 1.1, we will have:

$$\varphi_{1,1+2n}(x) = \begin{vmatrix} u = e^{-\frac{t^2}{2}} & du = -te^{-\frac{t^2}{2}}dt \\ dv = H_{2n}(t)dt & v = \frac{1}{2(2n+1)}H_{2n+1}(t)\end{vmatrix},$$

so

$$\psi_{1,1+2n}(x) = \frac{1}{\sqrt{(2n)!}\pi^{\frac14}2^n}\frac{e^{-\frac{x^2}{2}}}{2\,(2n+1)}H_{2n+1}(x) + \frac{1}{\sqrt{(2n)!}\pi^{\frac14}2^n}\int_{-\infty}^x te^{-\frac{t^2}{2}}H_{2n+1}(t)dt,$$

$$\psi_{1,1+2n}(x) = \begin{vmatrix} u = e^{-\frac{t^2}{2}} & du = \left(e^{-\frac{t^2}{2}} - t^2e^{-\frac{t^2}{2}}\right)dt \\ dv = H_{2n+1}(t)dt & v = \frac{1}{2(2n+2)}H_{2n+2}(t)\end{vmatrix}.$$

Then

$$\psi_{1,1+2n}(x) = \frac{1}{\sqrt{(2n)!}\pi^{\frac14}2^{n+1}}\frac{e^{-\frac{x^2}{2}}}{(2n+2)}H_{2n+1}(x) - \frac{1}{\sqrt{(2n)!}\pi^{\frac14}2^{n+1}}\frac{xe^{-\frac{x^2}{2}}}{(2n+2)}H_{2n+1}(x) + R_n(x),$$

where

$$R_n(x) = \frac{1}{2^{n+3}(2n+1)(n+1)\sqrt{(2n)!}\pi^{\frac14}}\int_{-\infty}^x \left(1-t^2\right)^{-\frac{t^2}{2}}H_{2n+2}(t)dt. \tag{2.1}$$

Theorem 2.1 (Third aim result)**.** *The following asymptotic formula holds:*

$$\psi_{1,1+2n}(x) = \frac{1}{\sqrt{(2n)!}\pi^{\frac14}2^{n+1}}\frac{e^{-\frac{x^2}{2}}}{(2n+2)}H_{2n+1}(x) - \frac{1}{\sqrt{(2n)!}\pi^{\frac14}2^{n+1}}\frac{xe^{-\frac{x^2}{2}}}{(2n+2)}H_{2n+1}(x) + R_n(x),$$

where $R_n(x)$ *is given by* (2.1) *and satisfies the estimate* $R_n(x) = o\left(\frac{1}{n}\right)$.

P r o o f. By the relationship between Hermite and Laguerre functions

$$H_{2n}(x) = C_n L_n^{-\frac{1}{2}}(x^2), \quad \text{where } C_n = (-1)^n 2^{2n},$$

we have

$$R_n(x) = \eta_n C_{n+1} \int_{-\infty}^{x} \left(1-t^2\right)^{-\frac{t^2}{2}} L_{n+1}^{-\frac{1}{2}}(t^2)dt,$$

where

$$\eta_n = \frac{1}{(2n+1)(n+1)\sqrt{(2n)!}\pi^{\frac{1}{4}}2^{n+3}}, \qquad L_n^{\alpha} \text{ is a Laguerre function.}$$

Introducing a new variable $u \doteq t^2$, we get

$$R_n(x) = \frac{\eta_n C_{n+1}}{2} \int_0^{x^2} \frac{1-u}{\sqrt{u}} e^{-\frac{u}{2}} L_{n+1}^{-\frac{1}{2}}(u)du.$$

To estimate the residue $R_n(x)$, we must consider two cases:

1. First case: $0 \le x^2 \le \frac{1}{n}$. For this case, we can use the weight estimate [9,10]

$$e^{-\frac{x}{2}} \left|L_n^{\alpha}(x)\right| \le c(\alpha) A_n^{\alpha}(x), \quad \alpha > -1,$$

and

$$A_n^{\alpha}(x) = \begin{cases} \theta_n^{\alpha}, & 0 \le x \le \frac{1}{\theta_n} \\ \theta_n^{\frac{\alpha}{2}-\frac{1}{4}} x^{-\frac{1}{2}-\frac{1}{4}}, & \frac{1}{\theta_n} < x < \frac{\theta_n}{2} \\ \left[\theta_n\left(\theta_n^{\frac{1}{3}} + |x-\theta_n|\right)\right]^{-\frac{1}{4}}, & \frac{\theta_n}{2} < x \le \frac{3\theta_n}{2} \\ e^{-\frac{1}{4}}, & \frac{3\theta_n}{2} < x, \end{cases}$$

where $\theta_n = \theta_n^{\alpha} = 4n + 2\alpha + 2$. So

$$e^{-\frac{x}{2}} \left|L_n^{-\frac{1}{2}}(x)\right| \le \frac{c\left(-\frac{1}{2}\right)}{4(n+1)+5},$$

then

$$|R_n(x)| \le \frac{\eta_n C_{n+1}}{2} \int_0^{x^2} \left|\frac{1-u}{\sqrt{u}}\right| \frac{\left|c\left(-\frac{1}{2}\right)\right|}{4n+5} du \le \frac{\eta_n C_{n+1}}{2} \int_0^{x^2} \left(\frac{1}{\sqrt{u}} + \sqrt{u}\right) du$$
$$< \frac{2}{4n+1} + \frac{2}{3}\frac{1}{(4n+1)^3}.$$

So $R_n(x) = o\left(\frac{1}{n}\right)$.

2. Second case: $\frac{1}{n} \le x^2 \le \omega$. We have

$$|R_n(x)| \le \frac{\eta_n C_{n+1}}{2} \int_0^{x^2} \left|\frac{1-u}{\sqrt{u}}\right| \left|L_n^{-\frac{1}{2}}(u)\right| du.$$

For this case, we use the following estimate:

$$e^{-\frac{x}{2}} x^{\frac{\alpha}{2}} L_n^{\alpha}(x) = N^{-\frac{\alpha}{2}} \frac{\Gamma(n+\alpha+1)}{n!} J_{\alpha}\left(2\,(Nx)^{\frac{1}{2}}\right) + o\left(n^{\frac{\alpha}{2}-\frac{3}{4}}\right),$$
$$N = n + \frac{\alpha+1}{2}, \quad x > 0, \quad \alpha > -1.$$
$$J_{\alpha}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + o\left(x^{-\frac{3}{2}}\right), \quad x \to +\infty.$$

We have

$$R_n(x) = \frac{\eta_n C_{n+1}}{2}\Big\{\int_0^{\frac{1}{n}} e^{-\frac{u}{2}}\,\frac{1-u}{\sqrt{u}}\,L_n^{-\frac{1}{2}}(u)du + \int_{\frac{1}{n}}^{x^2} e^{-\frac{u}{2}}\,\frac{1-u}{\sqrt{u}}\,L_n^{-\frac{1}{2}}(u)du\Big\},$$

so

$$\begin{aligned}|R_n(x)| &\le o\Big(\frac{1}{n}\Big) + \frac{\eta_n C_{n+1}}{2}\Big|\int_{\frac{1}{n}}^{x^2} e^{-\frac{u}{2}}\left(\frac{1-u}{\sqrt{u}}\right) L_n^{-\frac{1}{2}}(u)du\Big|.\\ &\le o\Big(\frac{1}{n}\Big) + \frac{\eta_n C_{n+1}}{2}\Big|\int_{\frac{1}{n}}^{x^2} \frac{e^{-\frac{u}{2}}}{\sqrt{u}} L_n^{-\frac{1}{2}}(u)du\Big| + \frac{\eta_n C_{n+1}}{2}\Big|\int_{\frac{1}{n}}^{x^2} e^{-\frac{u}{2}}\sqrt{u}L_n^{-\frac{1}{2}}(u)du\Big|.\end{aligned}$$

Calculate

$$\begin{aligned}A &= \Big|\int_{\frac{1}{n}}^{x^2} \frac{e^{-\frac{u}{2}}}{\sqrt{u}} L_n^{-\frac{1}{2}}(u)du\Big| = \Big|\int_{\frac{1}{n}}^{x^2} u^{-\frac{1}{4}}u^{-\frac{1}{4}}e^{-\frac{u}{2}}L_n^{-\frac{1}{2}}(u)du\Big|\\ &= \Big|\int_{\frac{1}{n}}^{x^2} u^{-\frac{1}{4}}\left(N^{\frac{1}{4}}\frac{\Gamma\left(n+\frac{3}{2}\right)}{(n+1)!}J_{-\frac{1}{2}}\left(2\sqrt{Nu}\right) + o\left(\frac{1}{n}\right)\right)du\Big|\\ &\le \Big|\int_{\frac{1}{n}}^{x^2} u^{-\frac{1}{4}}N^{\frac{1}{4}}\frac{\Gamma\left(n+\frac{3}{2}\right)}{(n+1)!}J_{-\frac{1}{2}}\left(2\sqrt{Nu}\right)du\Big| + \Big|\int_{\frac{1}{n}}^{x^2} o\left(\frac{1}{n}\right)u^{-\frac{1}{4}}du\Big|\\ &= N^{\frac{1}{4}}\frac{\Gamma\left(n+\frac{3}{2}\right)}{(n+1)!}\Big|\int_{\frac{1}{n}}^{x^2} u^{-\frac{1}{4}}J_{-\frac{1}{2}}\left(2\sqrt{Nu}\right)du\Big| + o\Big(\frac{1}{n}\Big)\Big|\int_{\frac{1}{n}}^{x^2} u^{-\frac{1}{4}}du\Big|.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}J_{-\frac{1}{2}}(x) &= \left(\frac{1}{\pi\sqrt{Nu}}\right)^{\frac{1}{2}}\cos\left(2\sqrt{Nu}+\frac{\pi}{4}-\frac{\pi}{4}\right) + o\Big(\left(2\sqrt{Nu}\right)^{-\frac{3}{2}}\Big)\\ &= \frac{1}{\sqrt{\pi}N^{\frac{1}{4}}u^{\frac{1}{4}}}\cos\left(2\sqrt{Nu}\right) + o\Big(\frac{1}{(Nu)^{\frac{3}{4}}}\Big).\end{aligned}$$

So

$$\varepsilon = \Big|\int_{\frac{1}{n}}^{x^2} u^{-\frac{1}{4}}J_{-\frac{1}{2}}\left(2\sqrt{Nu}\right)du\Big| = \frac{1}{\sqrt{\pi}N^{\frac{1}{4}}}\Big|\int_{\frac{1}{n}}^{x^2} u^{-\frac{1}{2}}\cos\left(2\sqrt{Nu}\right)du + \frac{1}{\sqrt{u}}\,o\Big(\frac{1}{(Nu)^{\frac{3}{4}}}\Big)\Big|.$$

Set a new variable as

$$t=\sqrt{Nu} \implies u=\frac{t^2}{N},\quad du=\frac{2t}{N}dt,\quad \sqrt{\frac{N}{u}}\le t\le x\sqrt{N}.$$

Then

$$\varepsilon \le \frac{2}{\sqrt{\pi}N^{\frac{3}{4}}}\int_{\sqrt{\frac{N}{u}}}^{x\sqrt{N}} |\cos 2t|\,dt + o\Big(\frac{1}{n}\Big) \le \frac{2}{\sqrt{\pi}N^{\frac{3}{4}}}\int_{\sqrt{\frac{N}{u}}}^{x\sqrt{N}} dt + o\Big(\frac{1}{n}\Big) = o\Big(\frac{1}{n}\Big),$$

and

$$A = \Big|\int_{\frac{1}{n}}^{x^2} \frac{e^{-\frac{u}{2}}}{\sqrt{u}}L_n^{-\frac{1}{2}}(u)du\Big| \le o\Big(\frac{1}{n}\Big).$$

For

$$B = \Big|\int_{\frac{1}{n}}^{x^2} e^{-\frac{u}{2}}\sqrt{u}L_n^{-\frac{1}{2}}(u)du\Big|,$$

in the same way, we get

$$B \leq o\left(\frac{1}{n}\right).$$

Then

$$R_n(x) \leq o\left(\frac{1}{n}\right) + o\left(\frac{1}{n}\right) + o\left(\frac{1}{n}\right) = o\left(\frac{1}{n}\right).$$

Finally, we have the desired estimate. □

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