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On the spectral properties and positivity of solutions of a periodic boundary value problem for a second-order functional differential equation

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О спектральных свойствах и положительности решений периодической краевой задачи для функционально-дифференциального уравнения второго порядка

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Abstract. For a functional-differential operator

$$\mathcal{L}u = (1/\rho)\left(-(pu')' + \int_0^l u(s)d_s r(x,s)\right)$$

with symmetry, the completeness and orthogonality of the eigenfunctions is shown. The positivity conditions of the Green function of the periodic boundary value problem are obtained.

Keywords: positive solutions; spectral properties

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Аннотация. Для функционально-дифференциального оператора

$$\mathcal{L}u = (1/\rho)\left(-(pu')' + \int_0^l u(s)d_s r(x,s)\right)$$

с симметрией показаны полнота и ортогональность собственных функций. Получены условия положительности функции Грина периодической краевой задачи

Ключевые слова: положительные решения; спектральные свойства

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1. Generalized string

Delay and deviation

Equation

$$-(pu')' + \int_0^l u(s)d_s r(x,s) = \rho(x)f(x),\ x \in [0,l],$$

was considered in [1–3] as a generalization of the string equation. In this regard, we mention book [4] devoted to the general theory of functional differential equations. Note, that usually this functional differential equation is presented as an delayed equation. This is due to the focus on applications to dynamic systems. The string oscillation problem has various generalizations, including the problems of quantum mechanics. In such problems, the presence of delay is not characteristic. Here, symmetry is more important, which is reflected both *in delay and in advance*. However, we can speak about delay and advance only conditionally, since we are talking about a spatial variable. Bearing in mind the problem of string vibrations, it is natural to consider the Sturm-Liouville boundary conditions

$$k_0 u(0) - pu'|_{x=0} = k_1 u(l) + pu'|_{x=l} = 0.$$

Here we are dealing with the eigenvalue problem under periodic boundary conditions

$$-(p(x)u')' + R(x)u - \int_0^l u(s)d_s r(x,s) = \lambda\rho(x)u \tag{1.1}$$

$$u(0) = u(l),\ pu'|_{x=0} = pu'|_{x=l}. \tag{1.2}$$

Function $r(x,\cdot)$ is assumed to be non-decreasing, and $r(x,0)=0$. In the second condition, expression pu' is considered as a quasi-derivative of function u.

Form (1.1) of the equation is not chosen by chance. In case $R(x) \geq r(x,l)$, it can be interpreted as the equation of a generalized loaded string. This mechanical interpretation allows us to predict the well-known properties of an ordinary string.

Assumptions. Notation

Under assumptions below, the boundary value problem $\mathcal{L}u = f$, (1.2), where the operator $\mathcal{L}$ is defined by the equality

$$\mathcal{L}u := \frac{1}{\rho(x)}\left(-(p(x)u')' + R(x)u - \int_0^l u(s)d_s r(x,s)\right), \tag{1.3}$$

has Fredholm property. Despite this, we use the variational method here, since our goal in particular is the properties of the eigenfunctions of Problem (1.1), (1.2).

the sign $:=$ means 'equal by definition'

It is assumed that p, ρ and $r(\cdot, s)$ (for all s) are measurable, $p(x) \geq c > 0$, $\rho(x) > 0$, $0 \leq x \leq l$, $r(x, \cdot)$ does not decrease for almost all x, $r(x, 0) = 0$. Let

$$\xi(x, y) := \int_0^x r(s, y)\, ds. \tag{1.4}$$

Assume that ξ is symmetris, that is, $\xi(x, y) = \xi(y, x)$, $x, y \in [0, l]$.

Let $I := [0, l]$. In the space $L_2(I, \rho)$ of functions with integrable square on I the scalar product is defined by

$$(f, g) := \int_0^l f(x) g(x)\, \rho(x)\, dx. \tag{1.5}$$

In this section assume that $R(x) \geq r(x, l)$, R is measurable and integrable on $[0, l]$,

$$\int_0^l (R(x) - r(x, l)) dx > 0. \tag{1.6}$$

Introduce the bilinear form

$$[u, v] := \int_0^l (p(x) u'(x) v'(x) + R(x) uv)\ dx - \int_0^l dx\, v(x) \int_0^l u(s) d_s r(x, s). \tag{1.7}$$

Let W be the set of absolutely continuous on $[0, l]$ functions satisfying the conditions $[u, u] < \infty$ and

$$u(0) = u(l). \tag{1.8}$$

Variational method

We follow the scheme of the variational method in [1]. Let us consider the equation in variational form with relation to u

$$\int_0^l (p(x) u'(x) v'(x) + R(x) uv)\ dx - \int_0^l dx\, v(x) \int_0^l u(s) d_s r(x, s) = \int_0^l f(x) v(x) \rho(x)\, dx. \tag{1.9}$$

It can be presented in the short form

$$[u, v] = (f, Tv),\ (\forall v \in W),\ f \in L_2(I, \rho). \tag{1.10}$$

Form $[u, v]$ is a scalar product (Lemma 1.1) in Hilbert space W of all absolutely continuous on I functions satisfying the boundary condition $u(0) = u(l)$. Let $T\colon W \to L_2(I, \rho)$ be defined by the equality $Tu(x) = u(x)$. The image $T(W)$ of the operator T is dense in $L_2(I, \rho)$. The equation (1.9) with respect to u has unique solution $u = T^* f$. It is equivalent to equation $\mathcal{L} u = f$, where $\mathcal{L} := (T^*)^{-1}$. Operator $\mathcal{L}$ can be represented by (1.3) (Lemma 1.3). Thus, eigenvalue problem (1.1), (1.2) has the short form

$$\mathcal{L} u = \lambda T u \tag{1.11}$$

Discreteness of spectrum of operator $\mathcal{L}$ is equivalent to compactness of the operator T. If T is compact, the eigenvalue problem has a system eigenfunctions u_n that forms an orthogonal basis in the space W. The system Tu_n forms an orthogonal basis in $L_2(I, \rho)$.

Theorem 1.1. *The equation (1.11) only has a nontrivial solution u_n in case of $\lambda = \lambda_n$, $n=1,2,\dots$. The sequence λ_n is non-decreasing sequence of positive numbers and $\lim \lambda_n = \infty$ (if the sequence is infinite one). The system u_n forms an orthogonal basis in the space W.*

Green function and positivity of solutions

From the previous it follows that equation

$$-(p(x)u')' + R(x)u - \int_0^l u(s)d_s r(x,s) = \rho(x)f(x) \tag{1.12}$$

has a unique solution in W for any $f \in L_2(I,\rho)$. But it can be considered on the more wide class of functions with nonzero boundary conditions. Let us consider the problem (1.12) with nonzero conditions

$$u(0) - u(l) = 0, \ (pu')|_{x=l} - (pu')|_{x=0} = \beta. \tag{1.13}$$

The Fredholm property of this problem can be shown as a generalization of problem (1.12), (1.2). However, we also refer to [4]. Since a homogeneous problem has only trivial solution, problem (1.12), (1.13) is uniquely solvable.

R e m a r k 1.1. We will use the fact [1] that the two-point boundary value problem

$$\mathcal{L}u = f, \ u(0) = a, \ u(l) = b \tag{1.14}$$

is uniquely solvable. Moreover, its solution is positive if $f \geq 0$, $a \geq 0$, $b \geq 0$, $\int_0^l f(x)\,dx + a + b > 0$ and if $a = b = 0$ then $(pu')(0) > 0$, $(pu')(l) < 0$.

Theorem 1.2. *If $\beta > 0$ the solution of the problem $\mathcal{L}u = 0$, (1.13) is positive on $[0,l]$.*

P r o o f. First, we note that the solution to problem $\mathcal{L}u = 0$, $u(0) = u(l) = 1$ is strictly positive (Remark 1.1). Let $v = 1 - u$. Then $\mathcal{L}v = \mathcal{L}(1) - \mathcal{L}u = R(x) - r(x,l) \geq 0$. Thus, $v \geq 0$ and $u \leq 1$. From here $(pu')(l) \geq 0$, $(pu')(0) \leq 0$. Since u is nontrivial solution, it is proportional to the solution of $\mathcal{L}u = 0$, (1.13). □

Theorem 1.3. *Suppose $f \geq\not\equiv 0$ and $u(x)$ is the solution of the problem (1.12) under boundary condition (1.2). Then $u(x) > 0$ on $[0,l]$.*

P r o o f. If $u(0) = u(l) = \alpha > 0$, then u is a solution to problem $\mathcal{L}u = f$, $u(0) = u(l) = \alpha$, which is positive (Remark 1.1). Suppose $\alpha \leq 0$. Let $v(x) = u(x) - \alpha$. Then $\mathcal{L}v = f - \alpha(R(x) - r(x,l)) \geq 0$. Thus v is solution to problem $\mathcal{L}v \geq 0$, $v(0) = v(l) = 0$. Thus $v \geq 0$. But u is not a constant. Thus $pv'(0) > 0$, $pv'(l) < 0$ that contradicts (1.2). □

R e m a r k 1.2. This statement also follows from the differential inequality theorem. This theorem 2.1 is considered below for more general case of equation. Put $v = 1$ in inequality (2.7). Then we have $R(x) - r(x,l) \geq\not\equiv 0$ in view of (1.6).

Auxiliary propositions

Lemma 1.1. *The set W with the inner product $[u,v]$ defined by (1.7) is a Hilbert space.*

P r o o f. If $u \neq \text{const}$ then $[u,u] \geq \int_0^l p(u')^2\,dx > 0$ because of the condition: $p > 0$ almost everywhere on $[0,l]$. If $u = \text{const} \neq 0$ then $[u,u] > 0$ in view of (1.6). Thus if $[u,u] = 0$ then $u = 0$, thence $[u,v]$ is a positive definite form.

To prove the completeness of W consider a Cauchy sequence $\{u_n\}$ in W. Then $\{u_n'\}$ is a Cauchy sequence in $L_2(I,p)$. Denote $\varphi = \lim u_n'$ in $L_2(I,p)$. Since $p \geq \text{const} > 0$,

$$\int_0^l \varphi(x)\,dx < \infty.$$

Let $u(x) = c + \int_0^x \varphi(x)dx$ where $c = \lim u_n(0)$ (it can be shown that $u_n(0)$ is a Cauchy sequence).

The remaining part of the proof can be realized in the same manner as it was done in [2]. □

Lemma 1.2. *$W \subset L_2(I,\rho)$ and the embedding operator $T\colon W \to L_2(I,\rho)$ is continuous.*

P r o o f. From the Bunyakovsky inequality

$$\left(\int_0^x u'(s)ds\right)^2 = \left(\int_0^x \frac{1}{\sqrt{p(s)}}\sqrt{p(s)}u'(s)ds\right)^2 \leq \int_0^x \frac{ds}{p(s)} \int_0^x p(s)u'(s)^2\,ds \leq \int_0^x \frac{ds}{p(s)} \cdot [u,u].$$

Hence

$$\int\limits_0^l u^2\rho\,dx = \int\limits_0^l \left(u(0) + \int_0^x u'(s)ds\right)^2 \rho\,dx = u(0)^2\int_0^l \rho\,dx + 2u(0)\int_0^l \left(\int_0^x u'(s)ds\right)\rho\,dx + \int_0^l \left(\int_0^x u'(s)ds\right)^2 \rho\,dx.$$

Since $[u,u] \geq \int_0^l qu^2\,dx$, it is easy to show that $u(0)^2 \leq C \cdot [u,u]$ where C doesn't depend on u. □

Lemma 1.3. *Operator $\mathcal{L}$ has representation (1.3), under boundary conditions (1.2).*

P r o o f. To find a representation of the operator $\mathcal{L} = (T^*)^{-1}$ we use the relation (1.10), $u = T^*f$ or $[u,v] = (f,Tv)$, $\forall v \in W$, or (1.9). Thus

$$\int_0^l p(x)u'v'\,dx = -\int_0^l R(x)uv\,dx + \int_0^l dx\,v(x)\int_0^l u(s)d_sr(x,s) + \int_0^l f(x)v(x)\rho(x)\,dx = -\int_0^l h'v\,dx,$$

where

$$h' = R(x)u - \int_0^l u(s)d_s r(x,s) - f(x)\rho(x).$$

Let $v(0) = v(l) = 0$ in

$$-\int_0^l h'v\,dx = -v(l)h(l) + v(0)h(0) + \int_0^l hv'\,dx.$$

We have from here $pu' = h + \text{const}$. Furthermore $-v(l)h(l) + v(0)h(0) = 0$ and $h(l) = h(0)$ because of $v(0) = v(l)$. Since $pu' = h + \text{const}$, pu' is absolutely continuous one (or is equivalent to). Thus

$$-(pu')' + Ru - \int_0^l u(s)\,d_s r(x,s) = f\rho$$

and $\mathcal{L}$ has representation (1.3).

Note that $(pu')(l) = (pu')(0)$ because of $h(l) = h(0)$. Thus u is a solution of the boundary value problem

$$-(pu')' + qu - \int_0^l (u(y) - u(x))d_y r(x,y) = \rho f, \tag{1.15}$$

$$u(0) = u(l), \qquad (pu')(0) = (pu')(l). \tag{1.16}$$

□

2. General case

In this second part we remove the restriction $R(x) \geq r(x,l)$. The condition of symmetry of function ξ defined by equality (1.4) is also removed.

In this case, the Green function can already change sign. We state a theorem on differential inequality, which serves as a necessary and sufficient condition for the positivity of the Green's function. The general scheme is simple, it is similar to how it was done in [1].

$$-(p(x)u')' + R(x)u - \int_0^l u(s)d_s r(x,s) = \rho(x)f(x). \tag{2.1}$$

Our equation (2.1) can be presented in the form $\mathcal{L}u = f$ where $\mathcal{L} = \mathcal{L}_0 - Q$,

$$\mathcal{L}_0 u := \frac{1}{\rho}\left(-(pu')' + R(x)u - \int_0^l u(s)d_s r_0(x,s)\right), \tag{2.2}$$

$$Qu := \int_0^l u(s)d_s q(x,s). \tag{2.3}$$

Function r_0 must be chosen so that inequality

$$R(x) \geq r_0(x,l) \tag{2.4}$$

holds. Then $q = r - r_0$. We require function q to be non-decreasing:

$$q(x,s) := \begin{cases} 0, & \text{if } r(x,s) \le R(x), \\ r(x,s) - R(x), & \text{otherwise.} \end{cases} \tag{2.5}$$

It is only necessary to make a small correction so that condition

$$\int_0^l (R(x) - r_0(x,l))dx > 0 \tag{2.6}$$

is satisfied. Assume that r_0 satistfies the symmetry condition

$$\xi_0(x,y) = \xi(y,x),$$

where $\xi_0(x,y) := \int_0^x r(s,y)ds$.

Positive solutions

The question of the positivity of the Green's function, and therefore of the positivity of the solutions of the boundary value problem, is solved by the necessary and sufficient condition in the theorem about differential inequality.

Theorem 2.1. *Let function $v \ge 0$ satisfy the differential inequality*

$$\mathcal{L}v \ge\not\equiv 0 \tag{2.7}$$

and boundary conditions (1.2). *Then the problem* (2.1),(1.2) *uniquely solvable and if $f \ge\not\equiv 0$ then $u(x) > 0,\ \ x \in [0,l]$.*

This theorem is proved in the same way as in [1].

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