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New method for the numerical solution of the Fredholm linear integral equation on a large interval

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Новый метод численного решения линейного интегрального уравнения Фредгольма на большом интервале

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Abstract. The traditional numerical process to tackle a linear Fredholm integral equation on a large interval is divided into two parts, the first is discretization, and the second is the use of the iterative scheme to approach the solutions of the huge algebraic system. In this paper we propose a new method based on constructing a generalization of the iterative scheme, which is adapted to the system of linear bounded operators. Then we don't discretize the whole system, but only the diagonal part of the system. This system is built by transforming our integral equation. As discretization we consider the product integration method and the Gauss–Seidel iterative method as iterative scheme. We also prove the convergence of this new method. The numerical tests developed show its effectiveness.

Keywords: Fredholm equation of the second kind; weakly singular kernel; large integration interval; Gauss–Seidel method; bounded operators matrix; product integration method

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Аннотация. Традиционное численное решение линейного интегрального уравнения Фредгольма на большом интервале делится на два этапа: первый — дискретизация, второй — использование итерационной схемы для приближения к решению алгебраической системы большой размерности (полученной на первом этапе). В этой статье мы предлагаем новый метод, основанный на построении обобщения итерационной схемы, которая адаптирована к системе линейных ограниченных операторов, при этом мы не дискретизируем всю систему, а только ее диагональную часть. Рассматриваемая система строится путем преобразования исходного интегрального уравнения. В качестве дискретизации мы рассматриваем метод интегрирования произведения, а в качестве итерационной схемы — итерационный метод Гаусса–Зайделя. Мы также анализируем сходимость этого нового метода. Численные тесты показывают его эффективность.

Ключевые слова: уравнение Фредгольма второго рода; слабо сингулярное ядро; большой интервал интегрирования; метод Гаусса–Зейделя; матрица ограниченных операторов; метод интегрирования произведений

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Introduction

Numerical approximation of linear Fredholm integral equations leads to linear algebraic systems. The size of the matrices obtained in the linear systems depends on the order of convergence. So, we have to solve a huge system to get a small error. However, this system can not be solved directly, so we use the Gauss–Seidel iterative scheme to approach its solutions. For this iterative method, many variants have been developed [1–6].

In this paper we propose a new method. First we make our integral equation into a system of the following form:

$$\begin{cases} \lambda u_1 &= T_{11}u_1 + T_{12}u_2 + \ldots + T_{1N}u_N + f_1, \\ \lambda u_2 &= T_{21}u_1 + T_{22}u_2 + \ldots + T_{2N}u_N + f_2, \\ \quad\vdots & \qquad\qquad\vdots \\ \lambda u_N &= T_{N1}u_1 + T_{N2}u_2 + \ldots + T_{NN}u_N + f_N, \end{cases} \tag{0.1}$$

where $\{T_{ij}\}_{1\leq i,j\leq N}$ is a family of bounded operators. Next we construct a generalization of the Gauss–Seidel method adapted to (0.1), after that we discretize only the diagonal part of this system to approach a solution of the initial equation.

In [7] and [8], a generalization of Jacobi's method adapted to the same system (0.1) has been constructed in order to approach a regular and a weakly singular Frehdolm integral equation, respectively. The numerical study of those papers presents very good results. In this paper, in a similar way, we construct a generalization of Gauss–Seidel method to approach a linear Fredholm integral equation of the second kind with weakly singular kernel defined on a large interval.

Let $X = C\left([0,\tau]\right)$ be the Banach space of continuous functions equipped with the norm

$$\forall x \in X \ \ \|x\|_X = \max_{0\leq t\leq\tau} |x(t)|\,,$$

where $[0,\tau]$ is a large interval of $\mathbb{R}$, i.e. $\tau >> 0$. Let $T : X \to X$ be the integral operator defined by

$$\forall x \in X \ \ Tx(t) = \int_0^\tau g(|s-t|)x(s)ds, \ \ t\in[0,\tau],$$

where $g : (0,\tau] \to \mathbb{R}$ is a weakly singular function in the following sense:

($\mathcal{H}$) the function g is continuous and decreasing on $(0,\tau]$, summable on $[0,\tau]$, $g(s)\geq 0$ for all $s\in(0,\tau]$, $\lim_{s\to 0+} g(s) = +\infty$.

Then $T : X \to X$ is a bounded operator [9], and the norm of T is given by

$$\|T\|_{BL(X)} = \sup_{\|x\|_X=1} \|Tx\|_X = \max_{0\leq t\leq\tau} \int_0^\tau g(|s-t|)ds = 2\int_0^{\tau/2} g(s)ds,$$

where $BL(X)$ is the Banach space of all bounded operators from X to itself. Let $\lambda \in \mathbb{C}^*$ be such that

$$|\lambda| = 2\mu \int_0^{\tau/2} g(s)ds, \quad \mu > 1,$$

then λ is in the resolvent set of T. By Neumann's theorem, we obtain that $(\lambda I - T)^{-1}$ exists, and

$$\left\|(\lambda I - T)^{-1}\right\|_{BL(X)} \leq \frac{1}{|\lambda| - \|T\|_{BL(X)}},$$

where I is the identity operator on X. Then the integral equation

$$\lambda u(t) = \int_0^{\tau} g(|s-t|)u(s)ds + f(t), \quad t \in [0, \tau], \tag{0.2}$$

has a unique solution $u \in X$ for every $f \in X$. Equation (0.2) is of great interest to mathematicians [9–11]. Our goal is to research this equation.

The paper is organized as follows. In Section 2, we introduce some notation and preliminary results. In Section 3, by using the previous results, we show how to formulate system (0.1). In Section 4, we treat our method of generalization of the Gauss–Seidel method in collocation with the product integration method. Finally, we give numerical results developed and compare our method with the conventional Gauss–Seidel method.

1. Notions and preliminary results

For $N \geq 2$, we define a subdivision of the interval $[0, \tau]$ by:

$$H = \frac{\tau}{N}, \; t_j = (j-1)H, \; 1 \leq j \leq N+1.$$

Let $\left\{\left(X_j, \|\cdot\|_j\right)\right\}_{j=1}^{N}$, $N \geq 2$, be a family of Banach spaces, where $X_j = C([t_j, t_{j+1}])$ is associated with the following norm:

$$\forall x \in X_j \quad \|x\|_j = \max_{t_j \leq t \leq t_{j+1}} |x(t)|.$$

For $1 \leq i, j \leq N$, we specify the Banach space $\mathbb{B}_{ij} = BL(X_j, X_i)$ of all bounded operators from X_j to X_i with the operator norm

$$\forall S \in \mathbb{B}_{ij} \quad \|S\|_{ij} = \sup_{\|x\|_j = 1} \|Sx\|_i.$$

Let $\widetilde{X}_N = \prod_{j=1}^{N} X_j$ be the product Banach space equipped with the norm

$$\forall Z = (z_1, ..., z_N) \in \widetilde{X}_N \quad \|Z\|_{\widetilde{X}_N} = \max_{1 \leq j \leq N} \|z_j\|_j.$$

Let $\widetilde{\mathbb{B}}_N = BL(\widetilde{X}_N)$ be the Banach space of all bounded operators from $\widetilde{X}_N$ to itself associated with the operator norm

$$\forall S \in \widetilde{\mathbb{B}}_N \quad \|S\| = \sup_{\|x\|_{\widetilde{X}_N} = 1} \|Sx\|_{\widetilde{X}_N}.$$

Let $\{T_{ij}\}_{1\le i,j\le N}$ be a family of operators defined by:

$$T_{ij}: C([t_j,t_{j+1}]) \to C([t_i,t_{i+1}]),\quad x \mapsto T_{ij}x(t) = \int_{t_j}^{t_{j+1}} g(|s-t|)x(s)ds,\ \ t\in[t_i,t_{i+1}].$$

It is clear that $T_{ij}\in\mathbb{B}_{ij}$ for all $1\le i,j\le N$, and

$$\|T_{ij}\|_{ij} = \max_{t\in[t_i,t_{i+1}]}\int_{t_j}^{t_{j+1}} g(|s-t|)ds.$$

For all $1\le i\le N$, $\|T_{ii}\|_{ii}<|\lambda|$, then $(\lambda I_{ii}-T_{ii})$ is a bijection on X_i, its inverse is a bounded linear operator (see [11]), and

$$\|(\lambda I_{ii}-T_{ii})^{-1}\|_{ii} \le \frac{1}{|\lambda|-\|T_{ii}\|_{ii}},$$

where I_{ii} is the identity operator on X_i.

Lemma 1.1. *For all $1\le i,j\le N$, we have:*

$$\|T_{ij}\|_{ij} = \begin{cases} 2\displaystyle\int_0^{H/2} g(s)ds, & \text{if} \quad i=j,\\ \displaystyle\int_{t_j-t_{i+1}}^{t_{j+1}-t_{i+1}} g(s)ds, & \text{if} \quad i<j,\\ \displaystyle\int_{t_i-t_{j+1}}^{t_i-t_j} g(s)ds, & \text{if} \quad i>j. \end{cases}$$

P r o o f. Consider the following function $G:[0,\tau]\to\mathbb{R}$ that will play an important role in the proof:

$$G(t) := \int_0^t g(s)ds.$$

Case 1: *i*=*j*. Let $y(t):[t_i,t_{i+1}]\to\mathbb{R}$ be defined by

$$y(t) := \int_{t_i}^{t_{i+1}} g(|s-t|)ds = \int_{t_i}^{t} g(t-s)ds + \int_t^{t_{i+1}} g(s-t)ds = G(t-t_i)+G(t_{i+1}-t).$$

The function $y(t)$ is symmetric with respect to $\frac{t_i+t_{i+1}}{2}$, and

$$y'(t) = g(t-t_i) - g(t_{i+1}-t).$$

Obviously, $y'(t)>0$ if $t_i<t<\frac{t_i+t_{i+1}}{2}$, and $y'(t)<0$ if $\frac{t_i+t_{i+1}}{2}<t<t_{i+1}$. Hence

$$\|T_{ii}\|_{ii} = \max_{t_i\le t\le t_{i+1}} y(t) = y\Big(\frac{t_i+t_{i+1}}{2}\Big) = 2\int_0^{H/2} g(s)ds.$$

Case 2: *i*<*j*. Let $y(t):[t_i,t_{i+1}]\to\mathbb{R}$ be defined by

$$y(t) := \int_{t_j}^{t_{j+1}} g(s-t)ds = G(t_{j+1}-t) - G(t_j-t).$$

Then

$$y'(t) = g(t_j - t) - g(t_{j+1} - t) > 0 \quad \text{for all} \quad t_i < t < t_{i+1}.$$

Hence

$$\|T_{ij}\|_{ij} = \max_{t_i \le t \le t_{i+1}} y(t) = y(t_{i+1}) = \int_{t_j - t_{i+1}}^{t_{j+1} - t_{i+1}} g(s)ds.$$

Case 3: $\boldsymbol{i > j}$. Let $y(t) : [t_i, t_{i+1}] \longrightarrow \mathbb{R}$ be defined by

$$y(t) := \int_{t_j}^{t_{j+1}} g(t - s)ds = G(t - t_j) - G(t - t_{j+1}).$$

Then

$$y'(t) = g(t - t_j) - g(t - t_{j+1}) < 0 \quad \text{for all} \quad t_i < t < t_{i+1}.$$

Hence

$$\|T_{ij}\|_{ij} = \max_{t_i \le t \le t_{i+1}} y(t) = y(t_i) = \int_{t_i - t_{j+1}}^{t_i - t_j} g(s)ds.$$

□

Theorem 1.1. *For integers $N \ge 2$, $1 \le i \le N$, consider the positive parameters $\underline{\gamma}_i$, $\overline{\gamma}_i$, $\beta_H(i, N)$ and β_* given by:*

$$\underline{\gamma}_i = \frac{\sum\limits_{j<i} \|T_{ij}\|_{ij}}{|\lambda| - \|T_{ii}\|_{ii}}, \qquad \overline{\gamma}_i = \frac{\sum\limits_{j>i} \|T_{ij}\|_{ij}}{|\lambda| - \|T_{ii}\|_{ii}},$$

$$\beta_H(i, N) = \underline{\gamma}_i + \overline{\gamma}_i, \qquad \beta_* = \max_{1 \le i \le N} \frac{\overline{\gamma}_i}{1 - \underline{\gamma}_i}.$$

We have

$$\beta_H(i, N) < 1 \quad \text{and} \quad \beta_* < 1.$$

P r o o f. Using the formulae obtained in Lemma 1.1, we get

$$\begin{aligned} \beta_H(i, N) &= \frac{\sum\limits_{j=1, j\ne i}^{N} \|T_{ij}\|_{ij}}{|\lambda| - \|T_{ii}\|_{ii}} = \frac{\sum\limits_{j=1}^{i-1} \|T_{ij}\|_{ij} + \sum\limits_{j=i+1}^{N} \|T_{ij}\|_{ij}}{|\lambda| - \|T_{ii}\|_{ii}} \\ &= \frac{\sum\limits_{j=1}^{i-1} \int_{t_i - t_{j+1}}^{t_i - t_j} g(s)ds + \sum\limits_{j=i+1}^{N} \int_{t_j - t_{i+1}}^{t_{j+1} - t_{i+1}} g(s)ds}{|\lambda| - \|T_{ii}\|_{ii}} \\ &= \frac{\int_0^{t_i} g(s)ds + \int_0^{\tau - t_{i+1}} g(s)ds}{2\mu \int_0^{\tau/2} g(s)ds - 2\int_0^{H/2} g(s)ds} = \frac{y_H(t_i)}{2\mu \int_0^{\tau/2} g(s)ds - 2\int_0^{H/2} g(s)ds}, \end{aligned}$$

where

$$y_H(t_i) := \int_0^{t_i} g(s)ds + \int_0^{\tau - t_{i+1}} g(s)ds = G(t_i) + G(\tau - t_{i+1}), \quad 1 \le i \le N.$$

The sequence $y_H(t_i)$ is symmetric with respect to $\frac{\tau}{2}$ or $\frac{\tau}{2} - H$, and

$$y'_H(t_i) = g(t_i) - g(\tau - t_{i+1}).$$

It is obvious, that $y'_H(t) > 0$ if $0 < t_i < \frac{\tau}{2} - H$, and $y'_H(t) < 0$ if $\frac{\tau}{2} < t_i < \tau$. Hence

$$\max_{1\leq i\leq N} y_H(t_i) = y_H\big(\frac{\tau}{2} - H\big) = y_H\big(\frac{\tau}{2}\big) = \int_0^{\tau/2} g(s)ds + \int_0^{\tau/2\ -H} g(s)ds.$$

As $N \geq 2$ then $H \leq \frac{\tau}{2}$, so we obtain

$$\begin{aligned}\beta_H(i,N) &\leq \frac{\max\limits_{1\leq i\leq N} y_H(t_i)}{2\mu\displaystyle\int_0^{\tau/2} g(s)ds - 2\int_0^{H/2} g(s)ds}, \\ &\leq \frac{\displaystyle\int_0^{\tau/2} g(s)ds + \int_0^{\tau/2\ -H} g(s)ds}{2\mu\displaystyle\int_0^{\tau/2} g(s)ds - 2\int_0^{H/2} g(s)ds} \leq \frac{1}{\mu} < 1.\end{aligned}$$

Finally, since $\beta_H(i,N) < 1$, it is clear that $\beta_* < 1$. □

2. Formulation of system (0.1)

In this section, we see how to formulate system (0.1) according to our integral equation. Let $\{u_j\}_{1\leq j\leq N}$ be a family of continuous functions such that

$$u_j \in X_j, \ \ \forall t \in [t_j, t_{j+1}] \ \ u_j(t) = u(t).$$

We have

$$\lambda u(t) = \int_0^{\tau} g(|s-t|)u(s)ds + f(t) = \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} g(|s-t|)u_j(s)ds + f(t), \quad t \in [0,\tau],$$

which is equivalent to the same system (0.1) described in the introduction:

$$\begin{cases} \lambda u_1(t) &= T_{11}u_1(t) + T_{12}u_2(t) + \ldots + T_{1N}u_N(t) + f_1(t), & t \in [t_1,t_2], \\ \lambda u_2(t) &= T_{21}u_1(t) + T_{22}u_2(t) + \ldots + T_{2N}u_N(t) + f_2(t), & t \in [t_2,t_3], \\ \quad\vdots & \qquad\qquad\qquad\vdots & \qquad\vdots \\ \lambda u_N(t) &= T_{N1}u_1(t) + T_{N2}u_2(t) + \ldots + T_{NN}u_N(t) + f_N(t), & t \in [t_N,t_{N+1}], \end{cases}$$

where

$$f_j \in X_j, \ \ \forall t \in [t_j, t_{j+1}] \ \ f_j(t) = f(t).$$

This system is equivalent to the following linear equation:

$$\lambda U = M_T U + F,$$

where $M_T : \widetilde{X}_N \to \widetilde{X}_N$ is the operator matrix defined by

$$M_T = \begin{pmatrix} T_{11} & T_{12} & \ldots & T_{1N} \\ T_{21} & T_{22} & \ldots & T_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ T_{N1} & T_{N2} & \ldots & T_{NN} \end{pmatrix},$$

$F=(f_1,...,f_N)$ is given in $\widetilde{X}_N$ and $U=(u_1,...,u_N)$ is to be found in the same space. It is clear that M_T is a bounded operator. We have

$$\|M_T\| = \max_{1\le i\le N}\sum_{j=1}^{N}\|T_{ij}\|_{ij} = \max_{1\le i\le N}\sum_{j=1}^{N}\max_{t\in[t_i,t_{i+1}]}\int_{t_j}^{t_{j+1}} g(|s-t|)ds$$
$$= \max_{0\le t\le \tau}\int_0^{\tau} g(|s-t|)ds < |\lambda|\,.$$

We use Neumann's theorem [12] to conclude that $(\lambda I_N - M_T)^{-1}$ exists and

$$\left\|(\lambda I_N - M_T)^{-1}\right\| \;\le\; \frac{1}{|\lambda| - \|M_T\|},$$

where I_N is the identity operator on $\widetilde{X}_N$. This assures the existence and uniqueness of the solution $U=(u_1,...,u_N)$ of the system (0.1) for all $F=(f_1,...,f_N)$ in $\widetilde{X}_N$.

3. Generalized Gauss–Seidel method

In this section, we construct a generalization of the Gauss–Seidel method suitable for our system (0.1).

3.1. Definition of an iterative sequence and its convergence

Consider the following iterative scheme:

$$\begin{cases} \lambda U^k = L_T U^k + (M_T - L_T)\,U^{k-1} + F, & k\ge 1,\\ U^0 \in \widetilde{X}_N, \end{cases}$$

where L_T is the lower triangular matrix part of M_T defined by

$$L_T = \begin{pmatrix} T_{11} & O_{12} & ... & O_{1N}\\ T_{21} & T_{22} & ... & O_{2N}\\ . & . & ... & .\\ T_{N1} & T_{N2} & ... & T_{NN} \end{pmatrix}.$$

For $1\le i,j\le N$, $O_{ij}: X_j \to X_i$ is the null operator, i.e. $\forall x\in X_j,\ O_{ij}x = 0_{X_i}$. We can write the precedent iterative scheme in a simple and clear formula: for $1\le i\le N$,

$$\begin{cases} \lambda u_i^k(t) = T_{ii}u_i^k(t) + \displaystyle\sum_{j=1}^{i-1} T_{ij}u_j^k(t) + \sum_{j=i+1}^{N} T_{ij}u_j^{k-1}(t) + f_i(t), & t\in[t_i,t_{i+1}], \quad k\ge 1,\\ u_i^0 \in X_i, \end{cases}$$

with

$$\sum_{j=1}^{0} T_{ij}u_j^k = \sum_{j=N+1}^{N} T_{ij}u_j^k = 0_{X_i}.$$

Our goal is to prove that $U^k \to U$ for $k\to\infty$.

Theorem 3.1. *We have*

$$\lim_{k\to\infty}\|U^k - U\|_{\widetilde{X}_N} = 0.$$

P r o o f. For all $1 \leq i \leq N$,

$$\lambda u_i = T_{ii} u_i + \sum_{j=1}^{i-1} T_{ij} u_j + \sum_{j=i+1}^{N} T_{ij} u_j + f_i,$$
$$(\lambda I_{ii} - T_{ii}) u_i = \sum_{j=1}^{i-1} T_{ij} u_j + \sum_{j=i+1}^{N} T_{ij} u_j + f_i,$$
$$u_i = (\lambda I_{ii} - T_{ii})^{-1} \sum_{j=1}^{i-1} T_{ij} u_j + (\lambda I_{ii} - T_{ii})^{-1} \sum_{j=i+1}^{N} T_{ij} u_j + (\lambda I_{ii} - T_{ii})^{-1} f_i.$$

In the same way, we get

$$u_i^k = (\lambda I_{ii} - T_{ii})^{-1} \sum_{j=1}^{i-1} T_{ij} u_j^k + (\lambda I_{ii} - T_{ii})^{-1} \sum_{j=i+1}^{N} T_{ij} u_j^{k-1} + (\lambda I_{ii} - T_{ii})^{-1} f_i.$$

Then

$$(u_i^k - u_i) = (\lambda I_{ii} - T_{ii})^{-1} \sum_{j=1}^{i-1} T_{ij} (u_j^k - u_j) + (\lambda I_{ii} - T_{ii})^{-1} \sum_{j=i+1}^{N} T_{ij} (u_j^{k-1} - u_j).$$

Therefore

$$\begin{aligned}
\|u_i^k - u_i\|_i &\leq \|(\lambda I_{ii} - T_{ii})^{-1}\|_{ii} \sum_{j=1}^{i-1} \|T_{ij}\|_{ij} \|u_j^k - u_j\|_j + \|(\lambda I_{ii} - T_{ii})^{-1}\|_{ii} \sum_{j=i+1}^{N} \|T_{ij}\|_{ij} \|u_j^{k-1} - u_j\|_j \\
&\leq \frac{\sum_{j=1}^{i-1} \|T_{ij}\|_{ij}}{|\lambda| - \|T_{ii}\|_{ii}} \|u_j^k - u_j\|_j + \frac{\sum_{j=i+1}^{N} \|T_{ij}\|_{ij}}{|\lambda| - \|T_{ii}\|_{ii}} \|u_j^{k-1} - u_j\|_j \\
&\leq \underline{\gamma}_i \|U^k - U\|_{\widetilde{X}_N} + \overline{\gamma}_i \|U^{k-1} - U\|_{\widetilde{X}_N}.
\end{aligned}$$

Let $i_m \in \mathbb{N}$ be such that

$$\|u_{i_m}^k - u_{i_m}\|_{X_{i_m}} = \|U^k - U\|_{\widetilde{X}_N}.$$

We obtain

$$\begin{aligned}
&\|U^k - U\|_{\widetilde{X}_N} \leq \underline{\gamma}_{i_m} \|U^k - U\|_{\widetilde{X}_N} + \overline{\gamma}_{i_m} \|U^{k-1} - U\|_{\widetilde{X}_N}, \\
&(1 - \underline{\gamma}_{i_m}) \|U^k - U\|_{\widetilde{X}_N} \leq \overline{\gamma}_{i_m} \|U^{k-1} - U\|_{\widetilde{X}_N}, \\
&\|U^k - U\|_{\widetilde{X}_N} \leq \frac{\overline{\gamma}_{i_m}}{1 - \underline{\gamma}_{i_m}} \|U^{k-1} - U\|_{\widetilde{X}_N}, \\
&\|U^k - U\|_{\widetilde{X}_N} \leq \beta_* \|U^{k-1} - U\|_{\widetilde{X}_N}.
\end{aligned}$$

Repeating this operation k times, we find that

$$\|U^k - U\|_{\widetilde{X}_N} \leq \beta_*^k \|U^0 - U\|_{\widetilde{X}_N}.$$

Now we use the fact that $\beta_* < 1$ to conclude the proof. □

To get an approximation of the solution u, we construct it, for the k-th iteration, using the following formula:

$$\forall t \in [t_i, t_{i+1}) \quad u(t) \approx u_i^k(t), \quad 1 \leq i \leq N,$$

and $u(b) = u_N^k(b)$.

3.2. Product integration method

In practice, for $1 \leq i \leq N$, $(\lambda I_{ii} - T_{ii})^{-1}$ can not be found exactly. For that, we need to approximate it using product integration method [10]. It will be easy, because, for $1 \leq i \leq N$, $[t_i, t_{i+1}]$ is not very large compared to $[0, \tau]$.

First of all, we study the following equation in order to explain the product integration method: for $1 \leq i \leq N$

$$\forall t \in [t_i, t_{i+1}] \quad \lambda u_i(t) = T_{ii} u_i(t) + g_i(t).$$

It is clear that $u_i \in X_i$ exists and is unique for all $g_i \in X_i$. For $n \geq 2$, $1 \leq i \leq N$, we define a subdivision of $[t_i, t_{i+1}]$ by

$$h_n = \frac{H}{n-1}, \quad s_{i,p} = t_i + (p-1)h_n, \quad 1 \leq p \leq n.$$

For $1 \leq i \leq N$, let $\{e_{i,p}(s)\}_{p=1}^n \subset X_i$ be such that for $2 \leq p \leq n-1$,

$$\begin{aligned}
e_{i,p}(s) &= \begin{cases} 1 - \dfrac{|s - s_{i,p}|}{h_n}, & s_{i,p-1} \leq s \leq s_{i,p+1} \\ 0, & otherwise. \end{cases} \\
e_{i,1}(s) &= \begin{cases} \dfrac{s_{i,2} - s}{h_n}, & s_{i,1} \leq s \leq s_{i,2} \\ 0, & otherwise. \end{cases} \\
e_{i,n}(s) &= \begin{cases} \dfrac{s - s_{i,n-1}}{h_n}, & s_{i,n-1} \leq s \leq s_{i,n} \\ 0, & otherwise. \end{cases}
\end{aligned}$$

Let $T_{ii,n} : X_i \to X_i$ be a linear operator defined by

$$\forall x \in X_i \quad T_{ii,n} x(t) = \sum_{p=1}^{n} w_{i,p}(t) x(s_{i,p}), \quad t \in [t_i, t_{i+1}],$$

with weights

$$w_{i,p}(t) = \int_{t_i}^{t_{i+1}} g(|s-t|) e_{i,p}(s) ds.$$

From [10], it follows that for $1 \leq i \leq N$, $T_{ii,n} \in \mathbb{B}_{ii}$ and

$$\|T_{ii,n}\|_{ii} \leq 2 \int_0^{H/2} g(s) ds.$$

Let us denote

$$\delta_H := 2 \int_0^{H/2} g(s) ds.$$

Theorem 3.1. *For all $1 \leq i \leq N$, for n large enough, $(\lambda I_{ii} - T_{ii,n})^{-1}$ exists and*

$$\left\| (\lambda I_{ii} - T_{ii,n})^{-1} \right\|_{ii} \leq \kappa,$$

where κ is a positive constant independent of i and n.

P r o o f. For all $1 \le i \le N$, using Lemma 4.1.2 [10], we get

$$\lim_{n\to\infty} \|(T_{ii} - T_{ii,n})T_{ii,n}\|_{ii} = 0.$$

From Theorem 4.1.2 [10], we obtain that $(\lambda I_{ii} - T_{ii,n})^{-1}$ exists for n large enough and

$$\left\|(\lambda I_{ii} - T_{ii,n})^{-1}\right\|_{ii} \le \frac{1 + \|(\lambda I_{ii} - T_{ii})^{-1}\|_{ii} \|T_{ii,n}\|_{ii}}{|\lambda| - \|(T_{ii} - T_{ii,n})T_{ii,n}\|_{ii}}.$$

But

$$\|T_{ii,n}\|_{ii} \le \delta_H, \quad \left\|(\lambda I - T_{ii})^{-1}\right\|_{ii} \le \frac{1}{|\lambda| - \|T_{ii}\|_{ii}} = \frac{1}{|\lambda| - \delta_H}.$$

Then

$$\lim_{n\to\infty} \frac{1 + \|(\lambda I_{ii} - T_{ii})^{-1}\|_{ii} \|T_{ii,n}\|_{ii}}{|\lambda| - \|(T_{ii} - T_{ii,n})T_{ii,n}\|_{ii}} \le |\lambda|^{-1} \left(1 + (|\lambda| - \delta_H)^{-1} \delta_H\right).$$

To conclude the proof, we take

$$\kappa = 1 + |\lambda|^{-1} \left(1 + (|\lambda| - \delta_H)^{-1} \delta_H\right).$$

□

For $1 \le i \le N$, for n large enough, let $u_{i,n} \in X_i$ be a unique solution of the following equation:

$$\lambda u_{i,n}(t) = T_{ii,n} u_{i,n}(t) + g_i(t), \quad t \in [t_i, t_{i+1}].$$

Theorem 3.2. *For $1 \le i \le N$, for n large enough,*

$$\|u_i - u_{i,n}\|_i \le \delta_H \kappa \omega (h_n, u_i),$$

where $\omega (h_n, u_i)$ is the modulus of continuity of $u_i(t)$ on $[t_i, t_{i+1}]$ defined by

$$\omega (h_n, u_i) = \max_{|s-t|\le h_n} |u_i(t) - u_i(s)|.$$

P r o o f. Follows from Theorem 4.2.1 in [10] and the fact that

$$u_i - u_{i,n} = (\lambda I_{ii} - T_{ii,n})^{-1} (T_{ii} - T_{ii,n}) u_i.$$

□

For $1 \le i \le N$, $u_{i,n}$ is calculated by the formula

$$\forall t \in [t_i, t_{i+1}] \quad u_{i,n}(t) = \frac{1}{\lambda}\Big(\sum_{p=1}^{n} w_{i,p}(t) x_p + g_i(t)\Big),$$

where $x = (x_1, ..., x_p)$ is a unique solution of the system

$$\lambda x = Ax + b, \quad A_{qp} = w_{i,p}(s_{i,q}), \quad b_q = g_i(s_{i,q}), \quad 1 \le q, p \le n.$$

We define the iterative scheme of the product integration version of the generalized Gauss–Seidel method corresponding to our integral equation by

$$\begin{cases} \lambda u_{i,n}^k(t) = T_{ii,n} u_{i,n}^k(t) + \sum_{j=1}^{i-1} T_{ij} u_{j,n}^k(t) + \sum_{j=i+1}^{N} T_{ij} u_{j,n}^{k-1}(t) + f_i(t), \quad t \in [t_i, t_{i+1}], \quad k \geq 1, \\ u_i^0 \in X_i, \end{cases}$$

for $1 \leq i \leq N$ and $n \geq 2$. For $k \geq 0$, we define $U_n^k = \left(u_{1,n}^k, u_{2,n}^k, \ldots, u_{N,n}^k\right) \in \widetilde{X}_N$. For technical reasons, we need to define $\hat{U}_n^k = \left(\hat{u}_{1,n}^k, \hat{u}_{2,n}^k, \ldots, \hat{u}_{N,n}^k\right) \in \widetilde{X}_N$: let for $1 \leq i \leq N$ and $n \geq 2$,

$$\begin{cases} \lambda \hat{u}_{i,n}^k(t) = T_{ii} \hat{u}_{i,n}^k(t) + \sum_{j=1}^{i-1} T_{ij} u_{j,n}^k(t) + \sum_{j=i+1}^{N} T_{ij} u_{j,n}^{k-1} + f_i(t), \quad t \in [t_i, t_{i+1}], \quad k \geq 1, \\ u_i^0 \in X_i. \end{cases}$$

Theorem 3.3. *For $k \geq 1$, $n \geq 2$, we have*

$$\|U_n^k - U\|_{\widetilde{X}_N} \leq \frac{\vartheta \delta_H \kappa}{1 - \beta_*} \max_{\substack{1 \leq i \leq N \\ 0 \leq l \leq k}} \omega\left(h_n, \hat{u}_{i,n}^l\right) + \beta_*^k \|U^0 - U\|_{\widetilde{X}_N},$$

where $\vartheta := \dfrac{|\lambda|}{|\lambda| - \|M_T\|}$.

P r o o f. We have, for $n \geq 2$,

$$\|U_n^k - U\|_{\widetilde{X}_N} \leq \|U_n^k - \hat{U}_n^k\|_{\widetilde{X}_N} + \|\hat{U}_n^k - U\|_{\widetilde{X}_N}.$$

But, for $1 \leq i \leq N$,

$$(\hat{u}_{i,n}^k - u_i) = (\lambda I_{ii} - T_{ii})^{-1} \sum_{j=1}^{i-1} T_{ij}(u_{j,n}^k - u_j) + (\lambda I_{ii} - T_{ii})^{-1} \sum_{j=i+1}^{N} T_{ij}(u_{j,n}^{k-1} - u_j).$$

Therefore

$$\begin{aligned} \|\hat{u}_{i,n}^k - u_i\|_i &\leq \|(\lambda I_{ii} - T_{ii})^{-1}\|_{ii} \sum_{j=1}^{i-1} \|T_{ij}\|_{ij} \|u_{j,n}^k - u_j\|_j \\ &+ \|(\lambda I_{ii} - T_{ii})^{-1}\|_{ii} \sum_{j=i+1}^{N} \|T_{ij}\|_{ij} \|u_{j,n}^{k-1} - u_j\|_j \\ &\leq \underline{\gamma}_i \|U_n^k - U\|_{\widetilde{X}_N} + \overline{\gamma}_i \|U_n^{k-1} - U\|_{\widetilde{X}_N}. \end{aligned}$$

Let $i_m \in \mathbb{N}$ be such that

$$\|\hat{u}_{i_m,n}^k - u_{i_m}^k\|_{X_{i_m}} = \|\hat{U}_n^k - U\|_{\widetilde{X}_N}.$$

We obtain

$$\|\hat{U}_n^k - U\|_{\widetilde{X}_N} \leq \underline{\gamma}_{i_m} \|U_n^k - U\|_{\widetilde{X}_N} + \overline{\gamma}_{i_m} \|U_n^{k-1} - U\|_{\widetilde{X}_N}.$$

Then

$$\begin{aligned}
\|U_n^k - U\|_{\widetilde{X}_N} &\le \|U_n^k - \hat{U}_n^k\|_{\widetilde{X}_N} + \underline{\gamma}_{i_m}\|U_n^k - U\|_{\widetilde{X}_N} + \overline{\gamma}_{i_m}\|U_n^{k-1} - U\|_{\widetilde{X}_N},\\
\|U_n^k - U\|_{\widetilde{X}_N} &\le \frac{1}{1-\underline{\gamma}_{i_m}}\|U_n^k - \hat{U}_n^k\|_{\widetilde{X}_N} + \frac{\overline{\gamma}_{i_m}}{1-\underline{\gamma}_{i_m}}\|U_n^{k-1} - U\|_{\widetilde{X}_N}\\
&\le \vartheta\delta_H\kappa \max_{1\le i\le N} \omega\left(h_n, \hat{u}_{i,n}^k\right) + \beta_*\|U_n^{k-1} - U\|_{\widetilde{X}_N}.
\end{aligned}$$

Repeating the last inequality,

$$\begin{aligned}
\|U_n^k - U\|_{\widetilde{X}_N} &\le \vartheta\delta_H\kappa \max_{1\le i\le N} \omega\big(h_n, \hat{u}_{i,n}^k\big) + \beta_*\Big(\vartheta\delta_H\kappa \max_{1\le i\le N} \omega\big(h_n, \hat{u}_{i,n}^{k-1}\big) + \beta_*\|U_n^{k-2} - U\|_{\widetilde{X}_N}\Big)\\
&\le \vartheta\delta_H\kappa \sum_{l=0}^{k-1} \beta_*^l \max_{1\le i\le N} \omega\left(h_n, \hat{u}_{i,n}^{k-l}\right) + \beta_*^k\|U^0 - U\|_{\widetilde{X}_N}\\
&\le \frac{\vartheta\delta_H\kappa}{1-\beta_*} \max_{\substack{1\le i\le N\\ 0\le l\le k}} \omega\left(h_n, \hat{u}_{i,n}^{l}\right) + \beta_*^k\|U^0 - U\|_{\widetilde{X}_N}.
\end{aligned}$$

□

4. Numerical Results

We illustrate the application of our numerical method by considering the following Fredholm integral equation of the second kind:

$$\lambda u(t) = \int_0^{100} \frac{u(s)}{\sqrt{|s-t|}}ds + f(t), \quad \lambda = 40\sqrt{2}, \quad t \in [0, 100],$$

where

$$f(t) = \lambda t^2 - \frac{2(3\times 10^4 + 400t + 8t^2)\sqrt{100-t}}{15} - \frac{16}{15}t^{5/2},$$

and the exact solution is $u(t) = t^2$ on $[0, 100]$. The kernel $g(s) = \frac{1}{\sqrt{s}}$ satisfies the hypothesis $(\mathcal{H})$. We mention that this equation is the same studied in [8].

In order to give a comprehensive view of the procedure of the generalized versions, we study this example by applying the following methods:

1. Generalized Gauss–Seidel method, our method described in this paper.
2. Conventional Gauss–Seidel method, the method described in [2], the latter applies the Gauss–Seidel iterative scheme to approach the huge matrix obtained after using the product integration method.
3. Generalized Jacobi method, the method described in [8].

We fix $H = 1$ and we take the null function as a starting point for our method and the null vector for the conventional Gauss–Seidel method. The stopping condition on the parameter k is fixed by

$$\|U_{new} - U_{old}\| \le 10^{-8}.$$

$E_{GGS}(h_n)$, $E_{CGS}(h_n)$ and $E_{GJ}(h_n)$ denote the absolute error obtained by using the Generalized Gauss–Seidel method, Conventional Gauss–Seidel method and Generalized Jacobi method, respectively. We vary now h_n to compare the results of the methods.

Table 1. Numerical results

h_n	E_{CGS}	E_{GGS}
0.250	9.86E-3	1.20E-3
0.125	2.47E-3	3.12E-4
0.050	3.97E-4	5.15E-5
0.025	1.01E-4	1.38E-5

Table 2. Numerical results

h_n	E_{GJ} (Results of [8])	k	E_{GGS}	k
0.250	1.20E-3	34	1.20E-3	20
0.125	3.12E-4	34	3.12E-4	20
0.050	5.15E-5	34	5.15E-5	20
0.025	1.38E-5	34	1.38E-5	20

Table (1) shows that the error committed by the two methods decreases with the decrease of h_n, but the error order of the generalized version of Gauss–Seidel method is smaller than the error order of the conventional version of Gauss–Seidel method. Furthermore, in Table (2) we can also see that the both generalized methods (Gauss-Seidel and Jacobi) give the same results, but the generalized Gauss-Seidel method is faster than the generalized Jacobi method. So, we confirm that our vision of generalization is reasonable.

5. Concluding remarks

We have constructed a generalization of the Gauss–Seidel iterative method for a system of linear operators. We used this new technique, in collocation with the product integration method, to approximate a solution of the Fredholm linear integral equation of the second kind with a weakly singular kernel on a large interval. The numerical tests show the efficiency of our new method compared to the classical Gauss–Seidel method.

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