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On λ-commuting and left (right) pseudospectrum and left (right) condition pseudospectrum of continuous linear operators on ultrametric Banach spaces

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Abstract. In this paper, we demonstrate some spectral properties of the λ-commuting of continuous linear operators on ultrametric Banach spaces and we introduce and study the operator equations $ASB = S$ and $AS = SB$. We give some properties of these operator equations. Some illustrative examples are provided. On the other hand, we introduce and study the left (right) pseudospectrum and the left (right) condition pseudospectrum of continuous linear operators on ultrametric Banach spaces. We prove that the left pseudospectra associated with various $\varepsilon > 0$ are nested sets and the intersection of all the left pseudospectra is the left spectrum. We give a relationship between the left (right) pseudospectrum and the left (right) condition pseudospectrum. Moreover, many results are proved concerning the left (right) pseudospectrum and the left (right) condition pseudospectrum of continuous linear operators on ultrametric Banach spaces.

Keywords: ultrametric Banach spaces, bounded linear operators, spectrum, left and right pseudospectrum

Mathematics Subject Classification: 47A10, 47S10.

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НАУЧНАЯ СТАТЬЯ

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О λ-коммутировании, левом (правом) псевдоспектре и левом (правом) условном псевдоспектре линейных непрерывных операторов на ультраметрических банаховых пространствах

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Аннотация. В работе мы демонстрируем некоторые спектральные свойства λ-коммутирования линейных непрерывных операторов в ультраметрических банаховых пространствах, а также изучаем операторные уравнения $ASB = S$ и $AS = SB$. Мы рассматриваем некоторые свойства этих операторных уравнений; приводим иллюстративные примеры. С другой стороны, мы вводим и изучаем левый (правый) псевдоспектр и левый (правый) условный псевдоспектр линейных непрерывных операторов в ультраметрических банаховых пространствах. Мы доказываем, что левые псевдоспектры, связанные с различными $\varepsilon > 0$, являются вложенными множествами, а пересечение всех левых псевдоспектров является левым спектром. Мы выявляем связь между левым (правым) псевдоспектром и левым (правым) условным псевдоспектром. Более того, доказываем еще ряд результатов, касающихся левого (правого) псевдоспектра и левого (правого) условного псевдоспектра линейных непрерывных операторов в ультраметрических банаховых пространствах.

Ключевые слова: ультраметрические банаховы пространства, линейные ограниченные операторы, спектр, левый и правый псевдоспектр

Для цитирования: *Эттайб Дж.* О λ-коммутировании, левом (правом) псевдоспектре и левом (правом) условном псевдоспектре линейных непрерывных операторов на ультраметрических банаховых пространствах // Вестник российских университетов. Математика. 2024. Т. 29. № 148. С. 494–516. https://doi.org/10.20310/2686-9667-2024-29-148-494-516 (In Engl., Abstr. in Russian)

1. Introduction and preliminaries

The classical theory of commutators was studied by H. Weyl [1] and J. von Neumann [2] and it played an important role in quantum mechanics [3–5]. In [6], C. R. Putnam collected some properties of the commutation of continuous linear operators in a Hilbert space over the field of complex numbers $\mathbb{C}$. Recently, many researchers studied and explored the operator equation $AS = \lambda SA$ where $\lambda \in \mathbb{C}\backslash\{0\}$, A and S are continuous linear operators on complex Hilbert spaces, see [7–9].

In ultrametric operator theory, the author [10] extended and studied the operator equation of the form $AS = \lambda SA$ where $\lambda \in \mathbb{K}\backslash\{0\}$, A and S are continuous linear operators on ultrametric Banach spaces over $\mathbb{K}$. He presented some spectral properties of λ-commuting operators on ultrametric Banach spaces over $\mathbb{K}$ and he gave an illustrative examples, see [10].

Recently, A. Ammar et al. [11] introduced and studied the pseudospectra of closed linear operators on ultrametric Banach spaces. On the other hand, A. Ammar et al. [12] introduced and studied the condition pseudospectra of continuous linear operators on ultrametric Banach spaces and gave some of its properties.

In [13], the author presented and studied the determinant spectrum, the M-determinant spectrum, and the C-trace pseudospectrum of ultrametric matrix pencils.

There are many studies on pseudospectra and condition pseudospectra of continuous linear operator pencils and λ-commuting of operators in ultrametric operator theory, see [14–17]. In Section 5., we consider the problem of finding the eigenvalues of the generalized eigenvalue problem of the form

$$P(\lambda)x = 0,$$

where $P(\lambda) = \sum_{k=0}^{m} \lambda^k A_k$, $A_k \in \mathcal{M}_n(\mathbb{K})$, $\lambda \in \mathbb{K}$, $x \in \mathbb{K}^n$ and $\mathcal{M}_n(\mathbb{K})$ is the space of all $n \times n$ matrices over $\mathbb{K}$. I is the identity matrix of $\mathcal{M}_n(\mathbb{K})$. If $C \in \mathcal{M}_n(\mathbb{K})$, the determinant of C is denoted by $\det(C)$ (for details on the space $\mathcal{M}_n(\mathbb{K})$ see [18] and [19]).

Throughout this paper, $\mathbb{Q}_p$ is the field of p-adic numbers, $\mathcal{E}$ is an ultrametric infinite-dimensional Banach space over a complete ultrametric valued field $\mathbb{K}$ with a non-trivial valuation $|\cdot|$ and $\mathcal{L}(\mathcal{E})$ denotes the set of all continuous linear operators on $\mathcal{E}$. Recall that $\mathbb{K}$ is called spherically complete if each decreasing sequence of balls in $\mathbb{K}$ has a non-empty intersection. For more details, see [20]. Let $S \in \mathcal{L}(\mathcal{E})$, $R(S)$, $N(S)$, S^*, $\sigma_p(S)$, $\sigma(S)$ and $\rho(S)$ denote the range, the kernel, the adjoint, the point spectrum, the spectrum and the resolvent set of S respectively [20].

The aim of this paper is to demonstrate some spectral properties of λ-commuting of continuous linear operators on ultrametric Banach spaces and we introduce and study the operator equations $AS = SB$ and $ASB = S$ for some $S \in \mathcal{L}(\mathcal{E})$. Moreover, some illustrative examples are provided. On the other hand, we introduce and study the left (right) pseudo-spectrum and the left (right) condition pseudospectrum of continuous linear operators on ultrametric Banach spaces. We obtain some results related to them. We continue by recalling some preliminaries.

Definition 1.1. [20] A field $\mathbb{K}$ is said to be ultrametric if it is endowed with an absolute value $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ such that

(i) $|\alpha| = 0$ if, and only if, $\alpha = 0$;

(ii) For all $\lambda, \alpha \in \mathbb{K}$, $|\lambda\alpha| = |\lambda||\alpha|$;

(iii) For each $\lambda, \alpha \in \mathbb{K}$, $|\lambda + \alpha| \leq \max\{|\lambda|, |\alpha|\}$.

D e f i n i t i o n 1.2. [20] Let $\mathcal{E}$ be a vector space over $\mathbb{K}$. A mapping $\|\cdot\| : \mathcal{E} \to \mathbb{R}_+$ is said to be an ultrametric norm if:

(i) For all $x \in \mathcal{E}$, $\|x\| = 0$ if and only if $x = 0$,

(ii) For any $x \in \mathcal{E}$ and $\lambda \in \mathbb{K}$, $\|\lambda x\| = |\lambda|\|x\|$,

(iii) For each $x, y \in \mathcal{E}$, $\|x + y\| \leq \max(\|x\|, \|y\|)$.

D e f i n i t i o n 1.3. [20] An ultrametric Banach space is a complete ultrametric normed space.

E x a m p l e 1.1. [20] Let $c_0(\mathbb{K})$ be the space of all sequences $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{K}$ such that $\lim_{i\to\infty} x_i = 0$. Then $c_0(\mathbb{K})$ is a vector space over $\mathbb{K}$ and

$$\|(x_i)_{i\in\mathbb{N}}\| = \sup_{i\in\mathbb{N}} |x_i|$$

is an ultrametric norm for which $(c_0(\mathbb{K}), \|\cdot\|)$ is an ultrametric Banach space.

Theorem 1.1. [21] *Let $\mathcal{E}$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. For each $x \in \mathcal{E}^* = \mathcal{E}\backslash\{0\}$, there exists $x^* \in \mathcal{E}^*$ such that $x^*(x) = 1$ and $\|x^*\| = \|x\|^{-1}$.*

D e f i n i t i o n 1.4. [20] An ultrametric Banach space $\mathcal{E}$ over $\mathbb{K}$ is called a free Banach space if there is a family $(x_i)_{i\in\mathcal{I}} \in \mathcal{E}$ indexed by a set $\mathcal{I}$ such that all $x \in \mathcal{E}$ is written in a unique fashion as $x = \sum_{i\in\mathcal{I}} \lambda_i x_i$ and $\|x\| = \sup_{i\in\mathcal{I}} |\lambda_i|\|x_i\|$. The family $(x_i)_{i\in\mathcal{I}}$ is called an orthogonal basis for $\mathcal{E}$. If, for each $i \in \mathcal{I}$, $\|x_i\| = 1$, hence $(x_i)_{i\in\mathcal{I}}$ is called an orthonormal basis of $\mathcal{E}$.

D e f i n i t i o n 1.5. [20] Let $\omega = (\omega_i)_i$ be a sequence of $\mathbb{K}^* = \mathbb{K}\backslash\{0\}$. We define $\mathcal{E}_\omega$ by

$$\mathcal{E}_\omega = \{x = (x_i)_i : \forall i \in \mathbb{N} \ x_i \in \mathbb{K}, \text{ and } \lim_{i\to\infty} |\omega_i|^{\frac{1}{2}}|x_i| = 0\},$$

and it is equipped with the norm

$$\forall x \in \mathcal{E}_\omega : \ x = (x_i)_i \ \ \|x\| = \sup_{i\in\mathbb{N}}(|\omega_i|^{\frac{1}{2}}|x_i|).$$

R e m a r k 1.1. [20]

(i) The space $(\mathcal{E}_\omega, \|\cdot\|)$ is an ultrametric Banach space.

(ii) If

$$\langle\cdot,\cdot\rangle : \mathcal{E}_\omega \times \mathcal{E}_\omega \longrightarrow \mathbb{K}, \quad (x, y) \mapsto \sum_{i=0}^{\infty} x_i y_i \omega_i,$$

where $x = (x_i)_i$ and $y = (y_i)_i$. Then the space $(\mathcal{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle)$ is called an ultrametric Hilbert space.

(iii) The orthogonal basis $\{e_i, i \in \mathbb{N}\}$ is called the canonical basis of $\mathbb{E}_\omega$ where for all $i \in \mathbb{N}$, $\|e_i\| = |\omega_i|^{\frac{1}{2}}$.

R e m a r k 1.2. [20] Let $\mathbb{K}=\mathbb{Q}_p$, if $p\equiv 1(mod\, 4)$, then $\mathrm{i}=\sqrt{-1}\in\mathbb{Q}_p$ and $\mathrm{i}^2=-1$.

D e f i n i t i o n 1.6. [20] Let $S\in\mathcal{M}_n(\mathbb{K})$. The spectrum $\sigma(S)$ of S is defined by

$$\sigma(S)=\{\lambda\in\mathbb{K}:\ S-\lambda I \text{ is not invertible}\}.$$

By Definition 6 of [13] (where $B=I$), we have the following:

D e f i n i t i o n 1.7. If $S\in\mathcal{M}_n(\mathbb{K})$ and $\varepsilon>0$. Then the ε-determinant spectrum $d_\varepsilon(S)$ of S is the following set:

$$d_\varepsilon(S)=\{\lambda\in\mathbb{K}:\ |\det(S-\lambda I)|\leq\varepsilon\}.$$

From Remark 2 of [13] (where $B=I$), we get

R e m a r k 1.3. Note that for each $S\in\mathcal{M}_n(\mathbb{K})$ and $\varepsilon>0$, $\sigma(S)\subseteq d_\varepsilon(S)$ and $d_0(S)=\sigma(S)$.

The λ-commuting of operators is defined as follows.

D e f i n i t i o n 1.8. [10] Let $A,B\in\mathcal{L}(\mathcal{E})$, A and B are called λ-commuting operators if $AB=\lambda BA$ for some $\lambda\in\mathbb{K}^*$.

E x a m p l e 1.2. [10] Let $\mathbb{K}=\mathbb{Q}_p$ with $p\equiv 1(mod\, 4)$, let A and B be defined on $\mathbb{Q}_p\times\mathbb{Q}_p$ respectively by

$$A=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \text{ and } B=\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}.$$

Then $AB=-BA$.

E x a m p l e 1.3. [10] Let $\lambda\in\mathbb{K}^*$, let A and B be defined on $\mathbb{K}^3$ by

$$A=\begin{pmatrix}1 & 0 & 0\\ 1 & \lambda & 0\\ 1 & \lambda & \lambda^2\end{pmatrix} \text{ and } B=\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}.$$

Then $AB=\lambda BA$.

E x a m p l e 1.4. [10] Let $\lambda\in\mathbb{K}$ such that $|\lambda|>1$ and let $A,B\in\mathcal{L}(c_0(\mathbb{K}))$ be given respectively by

$$B(x_1,x_2,x_3,\ldots)=\left(\frac{x_1}{\lambda},\frac{x_2}{\lambda^2},\frac{x_3}{\lambda^3},\ldots\right) \text{ for all } (x_1,x_2,x_3,\ldots)\in c_0(\mathbb{K})$$

and

$$\text{for all } n\geq 1,\quad Ae_n=e_{n+1},$$

where $(e_n)_{n\geq 1}$ is a base of $c_0(\mathbb{K})$. Hence $AB=\lambda BA$.

Let $A\in\mathcal{L}(\mathcal{E})$ be given, set $S_\lambda(A)=\{B\in\mathcal{L}(\mathcal{E}):\ AB=\lambda BA\}$. We collect some properties of λ-commuting operators.

P r o p o s i t i o n 1.1. [10] *Let* $A\in\mathcal{L}(\mathcal{E})$ *and* $\lambda\in\mathbb{K}$.

(i) *If* $B_1, B_2 \in S_\lambda(A)$, *hence* $B_1 + B_2 \in S_\lambda(A)$ *and* $B_1 B_2 \in S_{\lambda^2}(A)$;

(ii) *If* B *is invertible and* $\lambda \neq 0$, *then* $B^{-1} \in S_{\frac{1}{\lambda}}(A)$;

(iii) $S_\lambda(A)$ *is closed in the uniform operator topology*;

(iv) *If* $AB = \lambda BA$ *and* $AB \neq 0$, *then* $Ap(B) = p(\lambda B)A$ *where* p *is a non-constant polynomial.*

Proposition 1.2. [10] *Let* $A, B \in \mathcal{L}(\mathcal{E})$ *and* $\lambda \in \mathbb{Z}_p$ *such that* $AB = \lambda BA$, $AB \neq 0$ *and* $\|B\| < p^{\frac{1}{1-p}}$. *Then*

$$Ae^B = e^{\lambda B}A.$$

Proposition 1.3. [10] *If* $A, B \in \mathcal{L}(\mathcal{E})$, $\lambda \in \mathbb{K}^*$ *with* $AB = \lambda BA$, *hence* $\sigma_p(AB) = \lambda \sigma_p(BA)$.

Proposition 1.4. [10] *Let* $A, B \in \mathcal{L}(\mathcal{E})$, $\lambda \in \mathbb{K}^*$ *with* $AB = \lambda BA$. *Then* $\rho(AB) \subset \lambda \rho(BA)$. *Furthermore, for any* $\mu \in \rho(AB)$, $R(\mu, AB) = \lambda^{-1} R(\lambda^{-1}\mu, BA)$.

Proposition 1.5. [10] *Let* $A, B \in \mathcal{L}(\mathcal{E})$ *and* $\lambda \in \mathbb{K}^*$ *with* $AB = \lambda BA$. *Then*

(i) $N(AB) = N(BA)$;

(ii) $R(AB) = R(BA)$;

(iii) *For all* $\mu \in \mathbb{K}$, $N(AB - \mu) = N(BA - \lambda^{-1}\mu)$;

(iv) *For any* $\mu \in \mathbb{K}$, $R(AB - \mu) = R(BA - \lambda^{-1}\mu)$.

From Proposition 1.5, we conclude:

Theorem 1.2. [10] *If* $A, B \in \mathcal{L}(\mathcal{E})$ *and* $\lambda \in \mathbb{K}^*$ *with* $AB = \lambda BA$, *then*

$$\sigma_e(AB) = \lambda \sigma_e(BA).$$

For $A \in \mathcal{L}(\mathcal{E})$, set $r(A) = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}}$. We have the following proposition.

Proposition 1.6. [10] *Let* $A, B \in \mathcal{L}(\mathcal{E})$ *and* $\lambda \in \mathbb{K}$ *with* $|\lambda| = 1$ *and* $AB = \lambda BA$. *Then* $r(AB) \leq r(A)r(B)$.

We continue with the following definitions.

Definition 1.9. [20] Let $\mathcal{E}$ be a non-Archimedean Banach space over $\mathbb{K}$ and let $B \in \mathcal{L}(\mathcal{E})$, the spectrum $\sigma(B)$ of B is defined by

$$\sigma(B) = \{\mu \in \mathbb{K} : B - \mu I \text{ is not invertible}\},$$

the resolvent set of B is defined by $\rho(B) = \mathbb{K} \backslash \sigma(B)$.

Definition 1.10. [13] Let $B \in \mathcal{M}_n(\mathbb{K})$, the trace $Tr(B)$ of B is defined by $\sum_{i=1}^{n} b_{i,i}$ where for each $i \in \{1, \ldots, n\}$, $b_{i,i} \in \mathbb{K}$ are diagonal coefficients of B.

Proposition 1.7. [13] *Let* $B, C \in \mathcal{M}_n(\mathbb{K})$. *Then*

(i) *For any* $\lambda \in \mathbb{K}$, $Tr(B+\lambda C) = Tr(B) + \lambda Tr(C)$,

(ii) $Tr(BC) = Tr(CB)$.

We have:

Definition 1.11. [13] Let $B \in \mathcal{M}_n(\mathbb{K})$, $\varepsilon > 0$, the trace pseudospectrum $Tr_\varepsilon(B)$ of B is given by

$$Tr_\varepsilon(B) = \sigma(B) \cup \{\lambda \in \mathbb{K} : |Tr(B-\lambda I)| \leq \varepsilon\}.$$

The trace pseudoresolvent $Tr\rho_\varepsilon(B)$ of B is defined by

$$Tr\rho_\varepsilon(B) = \rho(B) \cap \{\mu \in \mathbb{K} : |Tr(B-\mu I)| > \varepsilon\}.$$

Lemma 1.1. [20] *Let* $S \in \mathcal{L}(\mathcal{E})$ *with* $\|S\| < 1$, *then* $\|(I-S)^{-1}\| \leq 1$.

2. λ-commuting of ultrametric operators

Similar to the proof of Proposition 1.6, we conclude:

Proposition 2.1. *Let* $A, B \in \mathcal{L}(\mathcal{E})$ *and* $\lambda \in \mathbb{K}$ *such that* $|\lambda| = 1$ *and* $AB = \lambda BA$. *Then* $r(BA) \leq r(A)r(B)$.

Question: In Proposition 2.1, if $|\lambda| \neq 1$, does $r(BA) \leq r(A)r(B)$ hold?

Definition 2.1. Suppose that $\|\mathcal{E}\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{L}(\mathcal{E})$, the approximate spectrum $\sigma_{ap}(A)$ of A is defined by

$$\sigma_{ap}(A) = \{\mu \in \mathbb{K} : \exists (x_n)_{n\in\mathbb{N}} \in \mathcal{E} \ \forall n \in \mathbb{N} \ \|x_n\| = 1 \text{ and } \lim_{n\to\infty} \|(A-\mu I)x_n\| = 0\}.$$

Proposition 2.2. *Suppose that* $\|\mathcal{E}\| \subseteq |\mathbb{K}|$. *If* $A, B \in \mathcal{L}(\mathcal{E})$, $\lambda \in \mathbb{K}^*$ *with* $AB = \lambda BA \neq 0$, *then* $\sigma_{ap}(AB) = \lambda\sigma_{ap}(BA)$.

Proof. Let $\mu \in \sigma_{ap}(AB)$, then there is $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that for each $n \in \mathbb{N}$, $\|x_n\| = 1$ and $\lim_{n\to\infty} \|(AB-\mu I)x_n\| = 0$. Since

$$\|(BA - \frac{\mu}{\lambda} I)x_n\| = \frac{\|(AB-\mu I)x_n\|}{|\lambda|}. \tag{2.1}$$

Then $\frac{\mu}{\lambda} \in \sigma_{ap}(BA)$ that is, $\mu \in \lambda\sigma_{ap}(BA)$. Similarly, if $\frac{\mu}{\lambda} \in \sigma_{ap}(BA)$ and using (2.1), we get $\mu \in \sigma_{ap}(AB)$. □

Lemma 2.1. *Let* $A, B \in \mathcal{L}(\mathcal{E})$, $\lambda \in \mathbb{K}$ *such that* $AB = \lambda BA \neq 0$. *Then for any* $n \in \mathbb{N}$, $A^n B = \lambda^n BA^n$.

Proof. Since $AB = \lambda BA \neq 0$. Then $A^2B = \lambda ABA = \lambda^2 BA^2$. One can see that for all $n \in \mathbb{N}$, $A^nB = \lambda^n BA^n$. □

Proposition 2.3. *Let* $A, B \in \mathcal{L}(\mathcal{E})$, $\lambda \in \mathbb{Z}_p$ *such that* $AB = \lambda BA$, $AB \neq 0$ *and* $\|A\| < p^{\frac{1}{1-p}}$. *Then*

$$e^A B = Be^{\lambda A}.$$

P r o o f. By $\|A\| < p^{\frac{1}{1-p}}$, we get e^A and $e^{\lambda A}$ exist. Since $AB = \lambda BA \neq 0$. Using Lemma 2.1, we conclude that $e^A B = Be^{\lambda A}$. □

In the finite-dimensional ultrametric Banach space, we obtain.

P r o p o s i t i o n 2.4. *If* $A, B \in \mathcal{M}_n(\mathbb{K})$ *are invertible matrices and* $\lambda \in \mathbb{K}^*$ *such that* $AB = \lambda BA$, *then* $\lambda^n = 1$.

P r o o f. From $\det(AB) = \lambda^n \det(BA)$ and $\det(BA) = \det(AB)$. We get $\lambda^n = 1$. □

From Proposition 4.2, we have the following:

Corollary 2.1. *If* $A, B \in \mathcal{M}_n(\mathbb{K})$ *are invertible matrices and* $\lambda \in \mathbb{K}^*$ *such that* $AB = \lambda BA$, *then* $|\lambda| = 1$.

P r o p o s i t i o n 2.5. *Let* $A, B \in \mathcal{M}_n(\mathbb{K})$ *and* $\lambda \in \mathbb{K}^*$ *with* $AB = \lambda BA$. *Then* $\mu \in d_\varepsilon(AB)$ *if and only if* $\frac{\mu}{\lambda} \in d_{\frac{\varepsilon}{|\lambda|^n}}(BA)$.

P r o o f. From $\det(AB - \mu I) = \det(\lambda BA - \mu I) = \lambda^n \det(BA - \frac{\mu}{\lambda} I)$ for $\mu \in \mathbb{K}$ and $\lambda \in \mathbb{K}^*$. Then $\mu \in d_\varepsilon(AB)$ if and only if $\frac{\mu}{\lambda} \in d_{\frac{\varepsilon}{|\lambda|^n}}(BA)$. □

P r o p o s i t i o n 2.6. *Let* $A, B \in \mathcal{M}_n(\mathbb{K})$ *and* $\lambda \in \mathbb{K}^*$ *with* $AB = \lambda BA$ *and* $AB \neq 0$. *If* $tr(AB) \neq 0$ *or* $tr(BA) \neq 0$, *then* $\lambda = 1$.

P r o o f. Since $tr(AB) = \lambda tr(BA) = \lambda tr(AB)$ and ($tr(AB) \neq 0$ or $tr(BA) \neq 0$), we get $\lambda = 1$. □

Let $\mathcal{E}$ is a free Banach space over $\mathbb{K}$, we set $\mathcal{L}_0(\mathcal{E}) = \{A \in \mathcal{L}(\mathcal{E}) : A^* \text{ exists}\}$.

P r o p o s i t i o n 2.7. [20] *If* $A, B \in \mathcal{L}_0(\mathcal{E}_\omega)$ *and* $\lambda \in \mathbb{K}$, *then*

(i) $(A + \lambda B)^* = A^* + \lambda B^*$.

(ii) $(AB)^* = B^* A^*$.

D e f i n i t i o n 2.2. [20] Let $A \in \mathcal{L}_0(\mathcal{E}_\omega)$. We have

(i) A is said to be selfadjoint if $A^* = A$;

(ii) A is said to be normal if $A^* A = AA^*$;

(iii) A is said to be unitary if $A^* A = AA^* = I$.

The following proposition describes some spectral properties of λ-commuting operators.

P r o p o s i t i o n 2.8. *Let* $A, B \in \mathcal{L}_0(\mathcal{E}_\omega)$ *and* $\lambda \in \mathbb{K}$ *with* $AB = \lambda BA \neq 0$. *If* A *is a selfadjoint, then* $ABB^* = BB^* A$ *and* ABB^* *is selfadjoint.*

P r o o f. If $A, B \in \mathcal{L}_0(\mathcal{E}_\omega)$ with $AB = \lambda BA$, then $(AB)^* = (\lambda BA)^*$. Hence

$$B^* A^* = \lambda A^* B^*. \tag{2.2}$$

Since A is a selfadjoint and by (2.2), we get $B^* A = \lambda AB^*$. On the other hand

$$ABB^* = \lambda BAB^* = \lambda \lambda^{-1} BB^* A = BB^* A,$$

and

$$(ABB^*)^* = (BB^*)^* A^* = BB^* A = ABB^*.$$

□

As the proof of Proposition 2.8, we get the following:

Proposition 2.9. *Let* $A, B \in \mathcal{L}_0(\mathcal{E}_\omega)$ *and* $\lambda \in \mathbb{K}$ *with* $AB = \lambda BA \neq 0$. *If* B *is a selfadjoint, then* $BAA^* = AA^*B$ *and* AA^*B *is selfadjoint.*

By Proposition 2.8 and Proposition 2.9, we conclude that:

Lemma 2.2. *Let* $A, B \in \mathcal{L}_0(\mathcal{E}_\omega)$ *and* $\lambda \in \mathbb{K}$ *with* $AB = \lambda BA \neq 0$. *If* A *and* B *are selfadjoint operators, then* $AB^2 = B^2A$ *and* $BA^2 = A^2B$.

Proposition 2.10. *Let* $A, B \in \mathcal{L}_0(\mathcal{E}_\omega)$ *and* $\lambda \in \mathbb{K}$ *with* $AB = \lambda BA \neq 0$ *and* $BA^2 \neq 0$. *If* A *and* B *are selfadjoint operators, then* $\lambda \in \{-1, 1\}$.

Proof. From $AB = \lambda BA$, we get $A^2B = \lambda^2 BA^2$. By Lemma 2.2, we have $A^2B = B^2A = \lambda^2 BA^2$. Since $BA^2 \neq 0$, we get $\lambda^2 = 1$. Then $\lambda \in \{-1, 1\}$. □

We give another proof of Proposition 2.10 without the condition $BA^2 \neq 0$.

Proposition 2.11. *Let* $A, B \in \mathcal{L}_0(\mathcal{E}_\omega)$ *and* $\lambda \in \mathbb{K}$ *with* $AB = \lambda BA \neq 0$. *If* A *and* B *are selfadjoint operators, then* $\lambda \in \{-1, 1\}$.

Proof. From A and B are selfadjoint operators, we get $(AB)^* = (\lambda BA)^*$, hence

$$BA = \lambda AB. \tag{2.3}$$

Using $AB = \lambda BA$ and (2.3), we get $AB = \lambda^2 AB$. Hence $\lambda^2 = 1$. Thus $\lambda \in \{-1, 1\}$. □

Proposition 2.12. *Let* $A, B \in \mathcal{L}(\mathcal{E}_\omega)$. *If there is an unitary operator* $U \in \mathcal{L}_0(\mathcal{E}_\omega)$ *with* $AB = UBA = BAU$, *then* $AB^2A = BA^2B$.

Proof. Since $AB = UBA = BAU$, we have

$$AB^2A = UBABA = BAUBA = BAAB = BA^2B.$$

□

Lemma 2.3. *Let* $A, B \in \mathcal{L}_0(\mathcal{E}_\omega)$ *be selfadjoint operators. If there is an unitary operator* $U \in \mathcal{L}_0(\mathcal{E}_\omega)$ *with* $AB = UBA$. *Then*

(i) U *and* U^* *commute with* AB;

(ii) U *and* U^* *commute with* BA.

Proof. (i) From $AB = UBA$, we have $BA = ABU^*$. Hence $AB = UBA = UABU^*$. Thus $ABU = UAB$ and $U^*AB = ABU^*$.

(ii) From (i), we get $U^*BA = (ABU)^* = (UAB)^* = BAU^*$. On the other hand $BAU = UU^*BAU = UBAU^*U = UBA$. □

Proposition 2.13. *Let* $A, B \in \mathcal{L}_0(\mathcal{E}_\omega)$ *be selfadjoint operators. If there is an unitary operator* $U \in \mathcal{L}_0(\mathcal{E}_\omega)$ *with* $AB = UBA$. *Then* $AB^2A = BA^2B$.

Proof. From Lemma 2.3, $UBA = BAU$. Therefore, $BA^2B = BAAB = BAUBA = UBABA = ABBA = AB^2A$. □

Lemma 2.4. *Let $A, B \in \mathcal{L}(\mathcal{E})$. If $AB^2 = B^2A$ and $BA^2 = A^2B$, then $AB^2A = BA^2B$.*

P r o o f. One can see that $AB^2A = B^2A^2$ and $BA^2B = B^2A^2$. Thus $AB^2A = BA^2B$. □

Question: Let $A, B \in \mathcal{L}_0(\mathcal{E})$ be selfadjoint operators. Is the converse of Lemma 2.4 hold?

E x a m p l e 2.1. Let $\mathbb{K} = \mathbb{Q}_p$ and let A and B be defined on $\mathbb{Q}_p^2$ by

$$A = \begin{pmatrix} a & 0 \\ b & \lambda a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where $\lambda a \in \mathbb{Q}_p \backslash \{0\}$. Then $AB = \lambda BA$.

E x a m p l e 2.2. Let $\lambda \in \mathbb{K}^*$, let A and B be defined on $\mathbb{K}^3$ by

$$A = \begin{pmatrix} a & 0 & 0 \\ b & \lambda a & 0 \\ c & \lambda b & \lambda^2 a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $AB = \lambda BA$.

Lemma 2.5. *Let $A, B \in \mathcal{L}_0(\mathcal{E}_\omega)$. If there is unitary operators $U, V \in \mathcal{L}_0(\mathcal{E}_\omega)$ with $AB = UB^*A^*$ and $BA = VA^*B^*$, then the following statements hold:*

(i) *AB commutes with U and U^*;*

(ii) *BA commutes with V and V^*.*

P r o o f. (i) From $AB = UB^*A^*$, we have $B^*A^* = ABU^*$. Hence $AB = UB^*A^* = UABU^*$. Thus $ABU = UAB$ and $U^*AB = ABU^*$.

(ii) By $BA = VA^*B^*$, we get $A^*B^* = BAV^*$. Thus $BA = VBAV^*$. Hence $BAV = VBA$ and $V^*BA = BAV^*$. □

Theorem 2.1. *If $A, B \in \mathcal{L}_0(\mathcal{E}_\omega)$ are selfadjoint operators and $\lambda \in \mathbb{K}^*$ with $AB = \lambda BA \neq 0$, then AB and BA are normal commuting operators.*

P r o o f. Set $S = AB$, hence $S^* = BA$. From $S = AB = \lambda BA$, we get $S = \lambda S^*$. Then $SS^* = \lambda (S^*)^2 = S^*S$. Thus AB is normal and $SS^* = S^*S$. Hence AB and BA are normal commuting operators. □

Theorem 2.2. *Let $A \in \mathcal{L}_0(\mathcal{E}_\omega)$ and let $B \in \mathcal{L}(\mathcal{E}_\omega)$ and $\lambda \in \mathbb{K}^*$ with $AB = \lambda BA \neq 0$, $A^*B = BA^*$ and $A^*A = I$ with $B \neq 0$. Then $\lambda = 1$.*

P r o o f. From $AB = \lambda BA$, hence $A^*AB = \lambda A^*BA$. Since $A^*B = BA^*$ and $A^*A = I$, we get $B = \lambda A^*BA = \lambda BA^*A = \lambda B$. Hence $(\lambda - 1)B = 0$. Since $B \neq 0$, we get $\lambda = 1$. □

P r o p o s i t i o n 2.14. *Let $A, B \in \mathcal{L}_0(\mathcal{E}_\omega)$. If there is unitary operators $U, V \in \mathcal{L}_0(\mathcal{E}_\omega)$ with $AB = UB^*A^*$ and $BA = VA^*B^*$. Then AB and BA are normal.*

P r o o f. By Lemma 2.5, we get $UB^*A^* = B^*A^*U$. Then

$$AB(AB)^* = ABB^*A^* = UB^*A^*B^*A^* = B^*A^*UB^*A^* = B^*A^*AB = (AB)^*AB$$

and

$$(BA)^*BA = A^*B^*BA = BAV^*BA = BABAV^* = BA(BA)^*.$$

□

Definition 2.3. [22] Let $A \in \mathcal{L}(\mathcal{E})$, A is said to be bounded below if for each $x \in \mathcal{E}$, $M\|x\| \leq \|Ax\|$ for some $M > 0$.

We have the following statement.

Theorem 2.3. *Let $A, B \in \mathcal{L}(\mathcal{E})$ and $\lambda \in \mathbb{K}^*$ such that $AB = \lambda BA \neq 0$. Then AB is bounded below if and only if A and B are bounded below.*

Proof. Suppose that AB is bounded below and $AB = \lambda BA \neq 0$. Then there is $M > 0$ with for each $x \in \mathcal{E}$,

$$M\|x\| \leq \|ABx\| \leq \|A\|\|Bx\|.$$

Hence B is bounded below. Since $M\|x\| \leq \|ABx\|$ for any $x \in \mathcal{E}$, and $AB = \lambda BA \neq 0$, it follows that for every $x \in \mathcal{E}$, $\frac{M}{|\lambda|\|B\|}\|x\| \leq \|Ax\|$. Consequently, A is bounded below.

Conversely, it is easy to see that if A and B are bounded below, then AB is bounded below. □

3. Some properties of ultrametric operator equations

In this section, let $A, B \in \mathcal{L}(\mathcal{E})$. We shall study the operator equations $AS = SB$ and $ASB = S$ for some $S \in \mathcal{L}(\mathcal{E})$. We continue with the following results.

Lemma 3.1. *Let $A, B, S \in \mathcal{L}(\mathcal{E})$ such that $AS = SB$ and $ASB = S$. If $A^2 - I$ or $I - B^2$ is invertible, then $S = 0$.*

Proof. From $AS = SB$ and $ASB = S$, we have $S = SB^2$. Then $S(I - B^2) = 0$. Since $I - B^2$ is invertible, we conclude that $S = 0$. Similarly, one can see that $(A^2 - I)S = 0$. From $A^2 - I$ is invertible, then $S = 0$. □

Further, $R(S)$ denotes the range of S dense in $\mathcal{E}$, i.e. $\overline{R(S)} = \mathcal{E}$.

Proposition 3.1. *Let $A, B, S \in \mathcal{L}(\mathcal{E})$ and $\overline{R(S)} = \mathcal{E}$ such that $AS = SB$ and $ASB = S$. Then $A^2 = I$.*

Proof. From $AS = SB$ and $ASB = S$, then $(A^2 - I)S = 0$. Hence $R(S) \subseteq N(A^2 - I)$. Since $\overline{R(S)} = \mathcal{E}$, we get $A^2 = I$. □

One can see the following:

Lemma 3.2. *Let $A, B, S \in \mathcal{L}(\mathcal{E})$ with $ASB = S$. If S is one to one, then B is one to one.*

Proof. It follows by S is one to one and $N(ASB) = N(S)$. □

Theorem 3.1. *Let $A, B \in \mathcal{L}(\mathcal{E})$ such that A is injective and $R(B)$ is dense. If $A^2S = SB^2$ and $A^3S = SB^3$, then $AS = SB$ for some $S \in \mathcal{L}(\mathcal{E})$.*

Proof. Set $U = AS$ and $V = SB$. Using $A^2S = SB^2$ and $A^3S = SB^3$. We get $AU = VB$ and $A^2U = VB^2$. Then $A(AU) = AVB = (UB)B$, thus $(AV - VB)B = 0$. By $R(B)$ is dense, we get $B \neq 0$ then $AV = VB$. From $AU = VB = AV$, we get $A(U - V) = 0$. From A is injective, then $U = V$. Hence $AS = SB$. □

Theorem 3.2. *Let* $A, B \in \mathcal{L}(\mathcal{E})$ *with* A *is injective and* $R(B)$ *is dense. If* $A^2SB^2 = S$ *and* $A^3SB^3 = S$, *then* $ASB = S$ *for some* $S \in \mathcal{L}(\mathcal{E})$.

Proof. From $A^2SB^2 = S$ and $A^3SB^3 = S$, then $A^2SB^2 = A^3SB^3$. Hence $A^2SB^2 - A^3SB^3 = 0$. Thus $A(A^2SB^2 - ASB)B = 0$. From A is injective and $R(B)$ is dense, we obtain that $A^2SB^2 - ASB = 0$. Thus $ASB = S$. □

4. Left (right) pseudospectum and left (right) condition pseudospectum of bounded linear operators on ultrametric Banach spaces

We introduce the following definitions.

Definition 4.1. Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$ and let $A \in \mathcal{L}(\mathcal{E})$.

(i) A is said to be left invertible if there exists $B \in \mathcal{L}(\mathcal{E})$ such that $BA = I$.

(ii) A is said to be right invertible if there exists $C \in \mathcal{L}(\mathcal{E})$ such that $AC = I$.

Definition 4.2. Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$. Let $A \in \mathcal{L}(\mathcal{E})$, the left spectrum $\sigma^l(A)$ of A is defined by

$$\sigma^l(A) = \{\lambda \in \mathbb{K} : A - \lambda I \text{ is not left invertible in } \mathcal{L}(\mathcal{E})\}.$$

Definition 4.3. Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$. Let $A \in \mathcal{L}(\mathcal{E})$, the right spectrum $\sigma^r(A)$ of A is defined by

$$\sigma^r(A) = \{\lambda \in \mathbb{K} : A - \lambda I \text{ is not right invertible in } \mathcal{L}(\mathcal{E})\}.$$

Definition 4.4. Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $A \in \mathcal{L}(\mathcal{E})$ and $\varepsilon > 0$, the left spectrum $\sigma^l_\varepsilon(A)$ of A is defined by

$$\sigma^l_\varepsilon(A) = \sigma^l(A) \cup \left\{\lambda \in \mathbb{K} : \inf\{\|C_l\| : C_l \text{ a left inverse of } A - \lambda I\} > \varepsilon^{-1}\right\},$$

with the convention $\inf\{\|C_l\| : C_l \text{ a left inverse of } A - \lambda I\} = \infty$ if $A - \lambda I$ is not left invertible.

Definition 4.5. Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $A \in \mathcal{L}(\mathcal{E})$ and $\varepsilon > 0$, the right spectrum $\sigma^r_\varepsilon(A)$ of A is defined by

$$\sigma^r_\varepsilon(A) = \sigma^r(A) \cup \left\{\lambda \in \mathbb{K} : \inf\{\|C_r\| : C_r \text{ a right inverse of } A - \lambda I\} > \varepsilon^{-1}\right\},$$

with the convention $\inf\{\|C_r\| : C_r \text{ a right inverse of } A - \lambda I\} = \infty$ if $A - \lambda I$ is not right invertible.

We obtain the following results.

Remark 4.1. From Definition 4.4 and Definition 4.5, we get

$$\sigma^l(A) \subset \sigma^l_\varepsilon(A) \subset \sigma_\varepsilon(A)$$

and

$$\sigma^r(A) \subset \sigma^r_\varepsilon(A) \subset \sigma_\varepsilon(A).$$

P r o p o s i t i o n 4.1. *Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $A \in \mathcal{L}(\mathcal{E})$ and $\varepsilon > 0$, we have*

(i) $\sigma^l(A) = \bigcap_{\varepsilon>0} \sigma^l_\varepsilon(A)$ *and* $\sigma^r(A) = \bigcap_{\varepsilon>0} \sigma^r_\varepsilon(A)$.

(ii) *For all* ε_1 *and* ε_2 *such that* $0 < \varepsilon_1 < \varepsilon_2$, $\sigma^l(A) \subset \sigma^l_{\varepsilon_1}(A) \subset \sigma^l_{\varepsilon_2}(A)$ *and* $\sigma^r(A) \subset \sigma^r_{\varepsilon_1}(A) \subset \sigma^r_{\varepsilon_2}(A)$.

P r o o f. (i) From Definition 4.4, for any $\varepsilon > 0$, $\sigma^l(A) \subset \sigma^l_\varepsilon(A)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \sigma^l_\varepsilon(A)$, hence for all $\varepsilon > 0$, $\lambda \in \sigma^l_\varepsilon(A)$. If $\lambda \notin \sigma^l(A)$, then

$$\lambda \in \{\lambda \in \mathbb{K} : \inf\{\|C_l\| : C_l \text{ a left inverse of } A - \lambda I\} > \varepsilon^{-1}\},$$

taking limits as $\varepsilon \to 0^+$, we get $\inf\{\|C_l\| : C_l \text{ a left inverse of } A - \lambda I\} = \infty$. Thus $\lambda \in \sigma^l(A)$. Similarly, we obtain $\sigma^r(A) = \bigcap_{\varepsilon>0} \sigma^r_\varepsilon(A)$.

(ii) For ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2$. Let $\lambda \in \sigma^l_{\varepsilon_1}(A)$, then

$$\inf\{\|C_l\| : C_l \text{ a left inverse of } A - \lambda I\} > \varepsilon_1^{-1} > \varepsilon_2^{-1},$$

hence $\lambda \in \sigma^l_{\varepsilon_2}(A)$. Similarly, we have $\sigma^r(A) \subset \sigma^r_{\varepsilon_1}(A) \subset \sigma^r_{\varepsilon_2}(A)$. □

P r o p o s i t i o n 4.2. *Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $A \in \mathcal{L}(\mathcal{E})$ and $\varepsilon > 0$. Then*

$$\bigcup_{C\in\mathcal{L}(\mathcal{E}):\|C\|<\varepsilon} \sigma^l(A + C) \subset \sigma^l_\varepsilon(A). \tag{4.1}$$

P r o o f. If $\lambda \in \bigcup_{C\in\mathcal{L}(\mathcal{E}):\|C\|<\varepsilon} \sigma^l(A + C)$. We argue by contradiction. Suppose that $\lambda \notin \sigma^l_\varepsilon(A)$, hence $\lambda \notin \sigma^l(A)$ and $\inf\{\|C_l\| : C_l \text{ a left inverse of } A - \lambda I\} \leq \varepsilon^{-1}$, thus $\|CC_l\| < 1$. Let D defined on $\mathcal{E}$ by

$$D = \sum_{n=0}^{\infty} C_l(-CC_l)^n.$$

One can see that D is well-defined and $D = C_l(I+CC_l)^{-1}$. Hence for all $y \in \mathcal{E}$, $D(I+CC_l)y = C_l y$. Set $y = (A - \lambda I)x$, we have for all $x \in \mathcal{E}$,

$$x = D(I + CC_l)(A - \lambda I)x = D(A - \lambda I + CC_l(A - \lambda I))x = D(A - \lambda I + C)x.$$

Hence $A + C - \lambda I$ is left invertible which is contradiction with $\lambda \in \bigcup_{C\in\mathcal{L}(\mathcal{E}):\|C\|<\varepsilon} \sigma^l(A + C)$. Thus, (4.1) holds. □

Theorem 4.1. *Let $\mathcal{E}$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|\mathcal{E}\| \subseteq |\mathbb{K}|$, let $A \in \mathcal{L}(\mathcal{E})$ and $\varepsilon > 0$. Then,*

$$\sigma^l_\varepsilon(A) = \bigcup_{C\in\mathcal{L}(\mathcal{E}):\|C\|<\varepsilon} \sigma^l(A + C).$$

P r o o f. According to the proposition 4.2 the embedding (inclusion) (4.1) is satisfied.

Conversely, suppose that $\lambda \in \sigma^l_\varepsilon(A)$. We discuss two cases.

First case: If $\lambda \in \sigma^l(A)$, we may set $C = 0$.

Second case: Assume that $\lambda \in \sigma^l_\varepsilon(A)$ and $\lambda \notin \sigma^l(A)$, then for all C_l a left inverse of $A - \lambda I$, we have $\|C_l\| > \frac{1}{\varepsilon}$. Hence, there exists $y \in \mathcal{E}\backslash\{0\}$ such that

$$\frac{\|C_l y\|}{\|y\|} > \frac{1}{\varepsilon}. \tag{4.2}$$

Set $y = (A - \lambda I)x$, then $C_l y = x$. From (4.2), we have $\|(A - \lambda I)x\| < \varepsilon\|x\|$. Since $\|\mathcal{E}\| \subseteq |\mathbb{K}|$, then there exists $c \in \mathbb{K}\backslash\{0\}$ such that $|c| = \|x\|$. Putting $z = c^{-1}x$, then $\|z\| = 1$, hence $\|(A - \lambda I)z\| < \varepsilon$. By Theorem 1.1, there exists $\phi \in \mathcal{E}^*$ such that $\phi(z) = 1$ and $\|\phi\| = \|z\|^{-1} = 1$. Define

$$\text{for all } y \in \mathcal{E},\ Cy = -\phi(y)(A - \lambda I)z.$$

Then $C \in \mathcal{L}(\mathcal{E})$ and $\|C\| < \varepsilon$, since for all $y \in \mathcal{E}$,

$$\|Cy\| = \|\phi(y)\|\|(A - \lambda I)z\| < \varepsilon\|y\|.$$

Furthermore, we have $(A - \lambda I + C)z = 0$. Thus $A - \lambda I + C$ is not left invertible. Consequently, $\lambda \in \bigcup_{C \in \mathcal{L}(\mathcal{E}): \|C\| < \varepsilon} \sigma^l(A + C)$. □

We continue with the following definitions.

D e f i n i t i o n 4.6. Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $A \in \mathcal{L}(\mathcal{E})$ and $\varepsilon > 0$, the left condition pseudospectrum $\Lambda^l_\varepsilon(A)$ of A is defined by

$$\Lambda^l_\varepsilon(A) = \sigma^l(A) \cup \big\{\lambda \in \mathbb{K} : \inf\{\|(A - \lambda I)\|\|D_l\| : D_l \text{ a left inverse of } A - \lambda I\} > \varepsilon^{-1}\big\},$$

with the convention $\inf\{\|(A - \lambda I)\|\|D_l\| :\ D_l$ a left inverse of $A - \lambda I\} = \infty$ if $A - \lambda I$ is not left invertible.

D e f i n i t i o n 4.7. Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $A \in \mathcal{L}(\mathcal{E})$ and $\varepsilon > 0$, the right condition pseudospectrum $\Lambda^r_\varepsilon(A)$ of A is defined by

$$\Lambda^r_\varepsilon(A) = \sigma^r(A) \cup \big\{\lambda \in \mathbb{K} : \inf\{\|A - \lambda I\|\|D_r\| :\ D_r \text{ a right inverse of } A - \lambda I\} > \varepsilon^{-1}\big\},$$

with the convention $\inf\{\|A - \lambda I\|\|D_r\| : D_r$ a right inverse of $A - \lambda I\} = \infty$ if $A - \lambda I$ is not right invertible.

We have the following results.

R e m a r k 4.2. From Definition 4.6 and Definition 4.7, we get

$$\sigma^l(A) \subset \Lambda^l_\varepsilon(A) \subset \Lambda_\varepsilon(A)$$

and

$$\sigma^r(A) \subset \Lambda^r_\varepsilon(A) \subset \Lambda_\varepsilon(A).$$

P r o p o s i t i o n 4.3. *Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $A \in \mathcal{L}(\mathcal{E})$ and $\varepsilon > 0$, we have*

(i) $\sigma^l(A) = \bigcap_{\varepsilon>0} \Lambda^l_\varepsilon(A)$ *and* $\sigma^r(A) = \bigcap_{\varepsilon>0} \Lambda^r_\varepsilon(A)$.

(ii) *For all* ε_1 *and* ε_2 *such that* $0 < \varepsilon_1 < \varepsilon_2$, $\sigma^l(A) \subset \Lambda^l_{\varepsilon_1}(A) \subset \Lambda^l_{\varepsilon_2}(A)$ *and* $\sigma^r(A) \subset \Lambda^r_{\varepsilon_1}(A) \subset \Lambda^r_{\varepsilon_2}(A)$.

Proof. (i) From Definition 4.6, for any $\varepsilon > 0$, $\sigma^l(A) \subset \Lambda^l_\varepsilon(A)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \Lambda^l_\varepsilon(A)$, hence for all $\varepsilon > 0$, $\lambda \in \Lambda^l_\varepsilon(A)$. If $\lambda \notin \sigma^l(A)$, then

$$\lambda \in \big\{\lambda \in \mathbb{K} : \inf\{\|A - \lambda I\|\|D_l\| : D_l \text{ a left inverse of } A - \lambda I\} > \varepsilon^{-1}\big\},$$

taking limits as $\varepsilon \to 0^+$, we get $\inf\{\|A - \lambda I\|\|D_r\| : D_l \text{ a left inverse of } A - \lambda I\} = \infty$. Hence $\lambda \in \sigma^l(A)$. Similarly, we obtain $\sigma^r(A) = \bigcap_{\varepsilon>0} \Lambda^r_\varepsilon(A)$.

(ii) For ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2$. Let $\lambda \in \Lambda^l_{\varepsilon_1}(A)$, then

$$\inf\{\|A - \lambda I\|\|D_l\| : D_l \text{ a left inverse of } A - \lambda I\} > \varepsilon_1^{-1} > \varepsilon_2^{-1},$$

hence $\lambda \in \Lambda^l_{\varepsilon_2}(A)$. Similarly, we have $\sigma^r(A) \subset \Lambda^r_{\varepsilon_1}(A) \subset \Lambda^r_{\varepsilon_2}(A)$. □

Proposition 4.4. *Let* $\mathcal{E}$ *be an ultrametric Banach space over* $\mathbb{K}$ *and let* $A \in \mathcal{L}(\mathcal{E})$ *and for every* $\varepsilon > 0$ *and* $\|A - \lambda I\| \neq 0$. *Then,*

(i) $\lambda \in \Lambda^l_\varepsilon(A)$ *if, and only if,* $\lambda \in \sigma^l_{\varepsilon\|A-\lambda I\|}(A)$.

(ii) $\lambda \in \sigma^l_\varepsilon(A)$ *if and only if* $\lambda \in \Lambda^l_{\frac{\varepsilon}{\|A-\lambda I\|}}(A)$.

Proof. (i) Let $\lambda \in \Lambda^l_\varepsilon(A)$, then $\lambda \in \sigma^l(A)$ or

$$\inf\{\|(A - \lambda I)\|\|C_l\| : C_l \text{ a left inverse of } A - \lambda I\} > \varepsilon^{-1}.$$

Hence $\lambda \in \sigma^l(A)$ or for all C_l a left invertible of $A - \lambda I$, $\|C_l\| > \frac{1}{\varepsilon\|(A-\lambda I)\|}$. Consequently, $\lambda \in \sigma^l_{\varepsilon\|A-\lambda I\|}(A)$. The converse is similar.

(ii) Let $\lambda \in \sigma^l_\varepsilon(A)$, then, $\lambda \in \sigma^l(A)$ or for all C_l a left inverse of $A - \lambda I$, $\|C_l\| > \varepsilon^{-1}$. Thus $\lambda \in \sigma^l(A)$ or for all C_l a left inverse of $A - \lambda I$, $\|(A - \lambda I)\|\|C_l\| > \varepsilon^{-1}\|(A - \lambda I)\|$. Then, $\lambda \in \Lambda^l_{\frac{\varepsilon}{\|A-\lambda I\|}}(A)$.

The converse is similar. □

One can see the following corollary.

Corollary 4.1. *Let* $\mathcal{E}$ *be an ultrametric Banach space over* $\mathbb{K}$, *let* $A \in \mathcal{L}(\mathcal{E})$ *and* $\varepsilon > 0$. *If* $\alpha, \beta \in \mathbb{K}$ *with* $\beta \neq 0$, *then* $\Lambda^l_\varepsilon(\beta A + \alpha I) = \alpha + \beta\Lambda^l_\varepsilon(A)$.

Proposition 4.5. *Let* $\mathcal{E}$ *be an ultrametric Banach space over* $\mathbb{K}$, *let* $A \in \mathcal{L}(\mathcal{E})$ *such that* $A \neq \lambda I$ *and* $C_A = \inf\{\|A - \lambda I\| : \lambda \in \mathbb{K}\}$ *and* $\varepsilon > 0$. *Then* $\sigma^l_\varepsilon(A) \subset \Lambda^l_{\frac{\varepsilon}{C_A}}(A)$.

Proof. Let $\mu \in \sigma^l_\varepsilon(A)$, then $\mu \in \sigma^l(A)$ or for all C_l a left inverse of $A - \mu I$, $\|C_l\| > \varepsilon^{-1}$. Since $\|A - \mu I\| \geq C_A > 0$. Then $\mu \in \sigma^l(A)$ or for all C_l a left inverse of $A - \mu I$, $\|A - \mu I\|\|C_l\| > \varepsilon^{-1}C_A$. Hence $\lambda \in \Lambda^l_{\frac{\varepsilon}{C_A}}(A)$. □

Lemma 4.1. *Let* $\mathcal{E}$ *be an ultrametric Banach space over* $\mathbb{K}$, *let* $A \in \mathcal{L}(\mathcal{E})$ *and* $\varepsilon > 0$. *If* $\lambda \in \Lambda^l(A)\backslash\sigma^l(A)$. *Then there exists* $x \in \mathcal{E}\backslash\{0\}$ *such that* $\|(A - \lambda I)x\| < \varepsilon\|A - \lambda I\|\|x\|$.

P r o o f. If $\lambda \in \Lambda^l(A)\backslash\sigma^l(A)$, then for all C_l a left inverse of $A-\lambda I$, we have

$$\|A-\lambda I\|\|C_l\| > \frac{1}{\varepsilon}.$$

Thus

$$\|C_l\| > \frac{1}{\varepsilon\|A-\lambda I\|}.$$

Then there exists $y \in \mathcal{E}\backslash\{0\}$ such that

$$\frac{\|C_l y\|}{\|y\|} > \frac{1}{\varepsilon\|A-\lambda I\|}. \tag{4.3}$$

Set $y=(A-\lambda I)x$, then $C_l y = x$. From (4.3), we have $\|(A-\lambda I)x\| < \varepsilon\|A-\lambda I\|\|x\|$. □

Theorem 4.2. *Let $\mathcal{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $A\in\mathcal{L}(\mathcal{E})$, $\lambda\in\mathbb{K}$ and $\varepsilon>0$. If there exists $C\in\mathcal{L}(\mathcal{E})$ with $\|C\|<\varepsilon\|A-\lambda I\|$ and $\lambda\in\sigma^l(A+C)$. Then, $\lambda\in\Lambda^l_\varepsilon(A)$.*

P r o o f. Assume that there exists $C\in\mathcal{L}(\mathcal{E})$ such that

$$\|C\|<\varepsilon\|A-\lambda I\| \text{ and } \lambda\in\sigma^l(A+C).$$

If $\lambda\notin\Lambda^l_\varepsilon(A)$, hence $\lambda\notin\sigma^l(A)$ and for each C_l a left inverse of $A-\lambda I$, $\|A-\lambda I\|\|C_l\|\leq\varepsilon^{-1}$. Consider D defined on $\mathcal{E}$ by

$$D=\sum_{n=0}^{\infty} C_l(-CC_l)^n.$$

Consequently $D=C_l(I+CC_l)^{-1}$. Hence for all $y\in\mathcal{E}$, $D(I+CC_l)y=C_l y$. Put $y=(A-\lambda I)x$, then

$$(\forall x\in\mathcal{E})\ D(A-\lambda I+C)x=x$$

Then $A-\lambda I+C$ is a left invertible which is a contradiction. Thus $\lambda\in\Lambda^l_\varepsilon(A)$. □

Set $\mathcal{C}_\varepsilon(\mathcal{E})=\{C\in\mathcal{L}(\mathcal{E}):\|C\|<\varepsilon\|A-\lambda I\|\}$, we have.

Theorem 4.3. *Let $\mathcal{E}$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|\mathcal{E}\|\subseteq|\mathbb{K}|$, let $A\in\mathcal{L}(\mathcal{E})$ and $\varepsilon>0$. Then,*

$$\Lambda^l_\varepsilon(A)=\bigcup_{C\in\mathcal{C}_\varepsilon(\mathcal{E})}\sigma^l(A+C).$$

P r o o f. By Theorem 4.2, we have $\bigcup_{C\in\mathcal{C}_\varepsilon(\mathcal{E})}\sigma^l(A+C)\subset\Lambda^l_\varepsilon(A)$. Conversely, assume that $\lambda\in\Lambda^l_\varepsilon(A)$. If $\lambda\in\sigma^l(A)$, we may put $C=0$. If $\lambda\in\Lambda^l_\varepsilon(A)$ and $\lambda\notin\sigma^l(A)$. By Lemma 4.1 and $\|\mathcal{E}\|\subseteq|\mathbb{K}|$, there exists $x\in\mathcal{E}\backslash\{0\}$ such that $\|x\|=1$ and $\|(A-\lambda I)x\|<\varepsilon\|A-\lambda I\|$.

By Theorem 1.1, there is $\varphi\in\mathcal{E}^*$ such that $\varphi(x)=1$ and $\|\varphi\|=\|x\|^{-1}=1$. Consider C on $\mathcal{E}$ defined by for all $y\in X$, $Cy=-\phi(y)(A-\lambda I)x$. Hence, $\|C\|<\varepsilon\|A-\lambda I\|$ and $D(C)=\mathcal{E}$. Moreover, for $x\in\mathcal{E}\backslash\{0\}$, $(A-\lambda I+C)x=0$. Then, $(A-\lambda I+C)$ is not left invertible. Consequently, $\lambda\in\bigcup_{C\in\mathcal{C}_\varepsilon(\mathcal{E})}\sigma^l(A+C)$. □

5. Determinant spectrum of non-Archimedean polynomial pencils

From Proposition 1 and Theorem 2 and Theorem 3 of [13], we get.

Theorem 5.1. *Let* $C \in \mathcal{M}_n(\mathbb{K})$. *Hence,*

(i) *If* $0 < \varepsilon_1 \leq \varepsilon_2$, $Tr_{\varepsilon_1}(C) \subset Tr_{\varepsilon_2}(C)$,

(ii) *If* $\beta \in \mathbb{K}$ *and* $\alpha \in \mathbb{K}\backslash\{0\}$, *hence* $Tr_\varepsilon(\alpha C + \beta I) = \alpha Tr_{\frac{\varepsilon}{|\alpha|}}(C) + \beta$,

(iii) *For any* $\alpha, \lambda \in \mathbb{K}$, *we have* $Tr_\varepsilon(\alpha I) = \left\{\lambda \in \mathbb{K} : |\lambda - \alpha| \leq \frac{\varepsilon}{|n|}\right\}$.

Theorem 5.2. *Let* $C, S, A \in \mathcal{M}_n(\mathbb{K})$ *and* $\varepsilon > 0$. *If* $S = ACA^{-1}$, *then* $Tr_\varepsilon(S) = Tr_\varepsilon(C)$.

Theorem 5.3. *Let* $C \in \mathcal{M}_n(\mathbb{K})$, *hence for any* $\varepsilon > 0$,

$$Tr_\delta(C) + B_f(0, \frac{\varepsilon}{|n|}) \subseteq Tr_\gamma(C)$$

with $\gamma = \max\{\varepsilon, \delta\}$, *if* $\delta < \varepsilon$, *we get*

$$Tr_\delta(C) + B_f(0, \frac{\varepsilon}{|n|}) \subseteq Tr_\varepsilon(C).$$

We have the following example.

Example 5.1. If

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Then for any $\varepsilon > 0$,

$$Tr_\varepsilon(A) = \{1, 2\} \cup \{\lambda \in \mathbb{Q}_p : |3 - 2\lambda|_p \leq \varepsilon\}.$$

By Definition 5 of [13], we get.

Definition 5.1. Let $C \in \mathcal{M}_n(\mathbb{K})$, $\varepsilon > 0$. Then the ε-trace set $tr_\varepsilon(C)$ of C is

$$tr_\varepsilon(C) = \{\lambda \in \mathbb{K} : |Tr(C - \lambda I)| \leq \varepsilon\}.$$

From Remark 1, Theorem 4, Proposition 2 and Proposition 3 of [13], we get.

Remark 5.1. For each $\varepsilon > 0$, $tr_\varepsilon(C) \subseteq Tr_\varepsilon(C)$.

Theorem 5.4. *If* $B, C \in \mathcal{M}_n(\mathbb{K})$. *Then, for any* $\varepsilon > 0$,

(i) $tr_\varepsilon(BC) = tr_\varepsilon(CB)$,

(ii) $tr_\varepsilon(B) + tr_\varepsilon(C) \subseteq tr_\varepsilon(B + C)$.

Proposition 5.1. *Let* $C \in \mathcal{M}_n(\mathbb{K})$, $\varepsilon > 0$, *if* $\lambda, \mu \in tr_\varepsilon(C)$ *and* $\alpha \in \mathbb{K}$ *with* $|\alpha| \leq 1$. *Then* $\alpha\mu + (1 - \alpha)\lambda \in tr_\varepsilon(C)$.

Proposition 5.2. *Let* $C \in \mathcal{M}_n(\mathbb{K})$, $\varepsilon > 0$ *with* $\|C\| < \varepsilon$. *If* $\lambda, \mu \in tr_\varepsilon(C)$, *then* $\lambda - \mu \in tr_\varepsilon(C)$.

The following propositions are valid.

P r o p o s i t i o n 5.3. *Let* $B \in \mathcal{M}_n(\mathbb{K})$, $\lambda \in \mathbb{K}$ *and* $\varepsilon > 0$. *If there exists* $C \in \mathcal{M}_n(\mathbb{K})$ *with* $|Tr(C)| \leq \varepsilon$ *and* $Tr(B - \lambda I - C) = 0$, *then* $\lambda \in tr_\varepsilon(B)$.

P r o o f. Since $Tr(B - \lambda I - C) = 0$ and $|Tr(C)| \leq \varepsilon$, hence $Tr(B - \lambda I) = Tr(C)$ and $|Tr(C)| \leq \varepsilon$, then $|Tr(B - \lambda I)| = |Tr(C)| \leq \varepsilon$. Hence $\lambda \in tr_\varepsilon(B)$. □

P r o p o s i t i o n 5.4. *Let* $A \in \mathcal{M}_n(\mathbb{K})$, $\lambda \in \mathbb{K}$ *and* $\varepsilon > 0$. *If there is* $C \in \mathcal{M}_n(\mathbb{K})$ *with* $|Tr(C)| > \varepsilon$ *and* $Tr(A - \lambda I - C) = 0$, *then* $\lambda \notin tr_\varepsilon(A)$.

P r o o f. Assume that there is $C \in \mathcal{M}_n(\mathbb{K})$ with $|Tr(C)| > \varepsilon$ and $Tr(A - \lambda I - C) = 0$. If $\lambda \in tr_\varepsilon(A)$, thus $|Tr(A - \lambda I)| = |Tr(C)| \leq \varepsilon$ which is contradiction with $|Tr(C)| > \varepsilon$ and $Tr(A - \lambda I - C) = 0$. □

P r o p o s i t i o n 5.5. *Let* $A \in \mathcal{M}_n(\mathbb{K})$, $\lambda \in \mathbb{K}$ *and* $\varepsilon > 0$. *If* $\lambda \notin tr_\varepsilon(A)$, *then there exists* $C \in \mathcal{M}_n(\mathbb{K})$ *with* $|Tr(C)| > \varepsilon$ *and* $Tr(A - \lambda I - C) = 0$.

P r o o f. If $\lambda \notin tr_\varepsilon(A)$, hence $|Tr(A - \lambda I)| > \varepsilon$. Set $C = \frac{Tr(A-\lambda I)}{n} I$. Thus $C \in \mathcal{M}_n(\mathbb{K})$ and $|Tr(C)| = |Tr(\frac{Tr(A-\lambda I)}{n} I)| = |\frac{Tr(A-\lambda I)}{n} Tr(I)| = |\frac{Tr(A-\lambda I)}{n} n| = |Tr(A - \lambda I)| > \varepsilon$. □

P r o p o s i t i o n 5.6. *Let* $A \in \mathcal{M}_n(\mathbb{K})$, $\lambda \in \mathbb{K}$ *and* $\varepsilon > 0$. *If* $\lambda \in tr_\varepsilon(A)$, *then there exists* $C \in \mathcal{M}_n(\mathbb{K})$ *with* $|Tr(C)| \leq \varepsilon$ *and* $Tr(A - \lambda I - C) = 0$.

P r o o f. If $\lambda \in tr_\varepsilon(A)$, hence $|Tr(A - \lambda I)| \leq \varepsilon$. Set $C = \frac{Tr(A-\lambda I)}{n} I$. Thus $C \in \mathcal{M}_n(\mathbb{K})$ and $|Tr(C)| = |Tr(\frac{Tr(A-\lambda I)}{n} I)| = |\frac{Tr(A-\lambda I)}{n} Tr(I)| = |\frac{Tr(A-\lambda I)}{n} n| = |Tr(A - \lambda I)| \leq \varepsilon$. □

From Definition 6 and Remark 2 of [13], we get.

D e f i n i t i o n 5.2. Let $C \in \mathcal{M}_n(\mathbb{K})$, $\varepsilon > 0$, the ε-determinant spectrum $d_\varepsilon(C)$ of C is the set

$$d_\varepsilon(C) = \{\lambda \in \mathbb{K} : |\det(C - \lambda I)| \leq \varepsilon\}.$$

R e m a r k 5.2. If $C \in \mathcal{M}_n(\mathbb{K})$, then for any $\varepsilon > 0$, $\sigma(C) \subseteq d_\varepsilon(C)$ and $d_0 = \sigma(C)$.

Using Proposition 4 of [13], we get.

P r o p o s i t i o n 5.7. *Let* $C \in \mathcal{M}_n(\mathbb{K})$. *Then for any* $\varepsilon > 0$,

(i) $\sigma(C) = \bigcap_{\varepsilon > 0} d_\varepsilon(C)$,

(ii) *For any* $0 < \varepsilon_1 \leq \varepsilon_2$, $d_{\varepsilon_1}(C) \subseteq d_{\varepsilon_2}(C)$.

We get:

E x a m p l e 5.2. Let

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Then for any $\varepsilon > 0$, $d_\varepsilon(C) = \{\mu \in \mathbb{Q}_p : |\mu(\mu - 2)|_p \leq \varepsilon\}$.

E x a m p l e 5.3. Let $a, b \in \mathbb{Q}_p^*$, $c \in \mathbb{Q}_p$ and

$$C = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Hence for any $\varepsilon > 0$, $d_\varepsilon(C) = \{\lambda \in \mathbb{Q}_p : |a - \lambda|_p |b - \lambda|_p \leq \varepsilon\}$.

Example 5.4. Let
$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$
Thus for any $\varepsilon > 0$, $d_\varepsilon(C) = \{\lambda \in \mathbb{Q}_p : |\lambda^2 - \lambda Tr(C) + \det(C)|_p \leq \varepsilon\}$.

Example 5.5. Let
$$C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$
Then for any $\varepsilon > 0$,
$$d_\varepsilon(C) = \{\lambda \in \mathbb{Q}_p : |\lambda^2|_p \leq \varepsilon\}.$$

We have the following propositions.

Proposition 5.8. *Let $D_1, D_2 \in \mathcal{L}(\mathbb{Q}_p^n)$ be two diagonal operators with for each $i \in \{1, \ldots, n\}$, $D_1 e_i = \lambda_i e_i$ and $D_2 e_i = \mu_i e_i$ with $\lambda_i, \mu_i \in \mathbb{Q}_p$, $\lambda_i \neq \lambda_{i+1}$ and $\mu_i \neq \mu_{i+1}$. Then $d_\varepsilon(D_1, D_2) = \{\mu \in \mathbb{Q}_p : |\lambda_1 - \mu\mu_1|_p \ldots |\lambda_n - \mu\mu_n|_p \leq \varepsilon\}$.*

Proof. For each $i \in \{1, \ldots, n\}$, $(D_1 - \lambda D_2)e_i = (\lambda_i - \lambda\mu_i)e_i$ where $(e_j)_{1 \leq j \leq n}$ is a basis of $\mathbb{Q}_p^n$. Hence, $|\det(D_1 - \lambda D_2)|_p = |\lambda_1 - \lambda\mu_1|_p \ldots |\lambda_n - \lambda\mu_n|_p$. Consequently for any $\varepsilon > 0$,
$$\begin{aligned} d_\varepsilon(D_1, D_2) &= \{\mu \in \mathbb{Q}_p : |\det(D_1 - \mu D_2)|_p \leq \varepsilon\} \\ &= \{\mu \in \mathbb{Q}_p : |\mu\mu_1 - \lambda_1|_p \ldots |\mu\mu_n - \lambda_n|_p \leq \varepsilon\}. \end{aligned}$$
□

Proposition 5.9. *Let $C \in \mathcal{M}_n(\mathbb{K})$ be invertible and $\lambda \in \mathbb{K}\backslash\{0\}$. Then for any $\varepsilon > 0$,*
$$\lambda \in d_\varepsilon(C) \quad \text{if and only if} \quad \lambda^{-1} \in d_{\frac{\varepsilon}{|\det(\lambda C)|}}(C^{-1}). \tag{5.1}$$

Proof. By virtue of the relation
$$\det(C - \lambda I) = \det(\lambda C(\lambda^{-1} - C^{-1})) = \det(\lambda C)\det(\lambda^{-1} - C^{-1}),$$
where $\lambda \neq 0$, (5.1) is satisfied. □

Proposition 5.10. *Let $B, C \in \mathcal{M}_n(\mathbb{K})$ with $\det(B) \neq 0$ and $\varepsilon > 0$. Then $d_\varepsilon(BC) = d_\varepsilon(CB)$.*

Proof. Since B is invertible, then
$$\begin{aligned} \det(BC - \lambda I) &= \det(B(C - \lambda B^{-1})) = \det(B)\det(C - \lambda B^{-1}) \\ &= \det(C - \lambda B^{-1})\det(B) = \det(CB - \lambda I). \end{aligned}$$
Then $\lambda \in d_\varepsilon(BC)$ if and only if $\lambda \in d_\varepsilon(CB)$. □

Now, we consider the problem of the eigenvalue of the polynomial pencil given by
$$P(\lambda)x = 0,$$
where $P(\lambda) = \sum_{k=0}^{n} \lambda^k A_k$ and $A_k \in \mathcal{M}_n(\mathbb{K})$ and $x \in \mathbb{K}^n$, we introduce the determinant pseudospectrum of polynomial pencils. Set $P(\lambda) = \sum_{k=0}^{n} \lambda^k A_k$ and $A_k \in \mathcal{M}_n(\mathbb{K})$, we have.

Definition 5.3. Let $P(\lambda) \in \mathcal{M}_n(\mathbb{K})$, the resolvent set $\rho(P(\lambda))$ of the polynomial pencil $P(\lambda)$ is

$$\rho(P(\lambda)) = \{\lambda \in \mathbb{K} : \ P(\lambda) \text{ is invertible}\},$$

the spectrum $\sigma(P(\lambda))$ of $P(\lambda)$ is $\mathbb{K}\backslash\rho(P(\lambda))$.

Definition 5.4. Let $P(\lambda) \in \mathcal{M}_n(\mathbb{K})$, $\varepsilon > 0$, the ε-determinant spectrum $d_\varepsilon(P(\lambda))$ of the polynomial pencil $P(\lambda)$ is defined by

$$d_\varepsilon(P(\lambda)) = \{\lambda \in \mathbb{K} : \ |\det(P(\lambda))| \leq \varepsilon\}.$$

Remark 5.3. From the Definition 5.4, if $P(\lambda) \in \mathcal{M}_n(\mathbb{K})$, then for any $\varepsilon > 0$, $\sigma(P(\lambda)) \subseteq d_\varepsilon(P(\lambda))$ and $d_0 = \sigma(P(\lambda))$.

Proposition 5.11. *If* $P(\lambda) \in \mathcal{M}_n(\mathbb{K})$, *then*

(i) $\sigma(P(\lambda)) = \bigcap_{\varepsilon>0} d_\varepsilon(P(\lambda))$,

(ii) *For all* $0 < \varepsilon_1 \leq \varepsilon_2$, *we have* $d_{\varepsilon_1}(P(\lambda)) \subseteq d_{\varepsilon_2}(P(\lambda))$.

Proof. (i) Obvious.
(ii) Let $0 < \varepsilon_1 \leq \varepsilon_2$ and $\lambda \in d_{\varepsilon_1}(P(\lambda))$. Then $|\det(P(\lambda))| \leq \varepsilon_1 \leq \varepsilon_2$. Hence $\lambda \in d_{\varepsilon_2}(P(\lambda))$. □

Example 5.6. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Set $P(\lambda) = \lambda^2 A^2 - I$. Then for any $\varepsilon > 0$,

$$d_\varepsilon(P(\lambda)) = \{\lambda \in \mathbb{Q}_p : \ |(2\lambda - 1)(2\lambda + 1))|_p \leq \varepsilon\}.$$

Let $A, B, C \in \mathcal{M}_n(\mathbb{K})$. We consider $P(\lambda) = \lambda^2 A + \lambda B + C$. For all $\lambda \in \rho(P(\lambda))$, $R(\lambda, P) = (\lambda^2 A + \lambda B + C)^{-1}$.

Proposition 5.12. *Let* $A, B, C \in \mathcal{M}_n(\mathbb{K})$. *If the inclusions* $\lambda \in \rho(P)$ *and* $\mu \in \mathbb{K}$ *with* $\|R(\lambda, P)((\lambda^2 - \mu^2)A + (\lambda - \mu)B)\| < 1$, *then* $\mu \in \rho(P)$ *and* $\|R(\mu, P)\| \leq \|R(\lambda, P)\|$.

Proof. Because

$$\begin{aligned} \mu^2 A + \mu B + C &= \lambda^2 A + \lambda B + C - ((\lambda^2 - \mu^2)A + (\lambda - \mu)B) \\ &= (\lambda^2 A + \lambda B + C)\left(I - R(\lambda, P)((\lambda^2 - \mu^2)A + (\lambda - \mu)B)\right), \end{aligned}$$

and $\|R(\lambda, P)((\lambda^2 - \mu^2)A + (\lambda - \mu)B)\| < 1$, there is $R(\mu, P)$ and $\|R(\mu, P)\| \leq \|R(\lambda, P)\|$. □

The next theorem presents the perturbation of operators.

Theorem 5.5. *Let* $\mathcal{E}$ *be a non-Archimedean Banach space over* $\mathbb{K}$, *let* $B, C \in \mathcal{L}(\mathcal{E})$. *Let* $\mu \in \rho(B, C)$ *and* $\lambda \in \mathbb{K}$ *with* $|\lambda - \mu| < \|R(\mu, B, C)C\|^{-1}$, *then* $\lambda \in \rho(B, C)$ *and* $\|R(\lambda, B, C)\| \leq \|R(\mu, B, C)\|$ *where* $R(\lambda, B, C) = (B - \lambda C)^{-1}$.

P r o o f. Let $\mu \in \rho(B,C)$, we have:

$$B - \lambda C = B - \mu C + \mu C - \lambda C = (B - \mu C)\big(I - (\lambda - \mu)R(\mu, B, C)C\big).$$

Since $\lambda \in \mathbb{K}$ with $|\lambda - \mu| < \|R(\mu, B, C)C\|^{-1}$, by Lemma 1.1,

$$\big(I - (\lambda - \mu)R(\mu, B, C)C\big)^{-1} \text{ and } \|\big(I - (\lambda - \mu)R(\mu, B, C)C\big)^{-1}\| \leq 1.$$

Then $\lambda \in \rho(B,C)$ and $R(\lambda, B, C) = \big(I - (\lambda - \mu)R(\mu, B, C)C\big)^{-1}R(\mu, B, C)$. Thus $\lambda \in \rho(B,C)$ and $\|R(\lambda, B, C)\| \leq \|R(\mu, B, C)\|$. □

From Theorem 5.5, we have.

Corollary 5.1. *$\sigma(B,C)$ is closed in $\mathbb{K}$.*

From the results of M. Vishik [23] for $C = I$, there is a nonanalytic resolvent of an operator, for that we assume that application $\lambda \in \rho(B,C) \mapsto R(\lambda, B, C) = (B - \lambda C)^{-1}$ is analytic on $\rho(B,C)$.

Theorem 5.6. *Let $\mathcal{E}$ be a non-Archimedean Banach space over an algebraically closed field $\mathbb{K}$, let $B, C \in \mathcal{L}(\mathcal{E})$ with $R(\lambda, B, C)$ is analytic on $\rho(B,C)$. Then*

$$\frac{d}{d\lambda}R(\lambda, B, C) = R(\lambda, B, C)CR(\lambda, B, C). \tag{5.2}$$

P r o o f. Let $\lambda \in \rho(B,C)$, let $\mu \in \mathbb{K}$ with $|\lambda - \mu| < \|R(\lambda, B, C)C\|^{-1}$, by Theorem 5.5, we have:

$$\begin{aligned} R(\mu, B, C) &= \big(I - (\mu - \lambda)R(\lambda, B, C)C\big)^{-1}R(\lambda, B, C) \\ &= \sum_{k=0}^{\infty}\big((\mu - \lambda)R(\lambda, B, C)C\big)^{k}R(\lambda, B, C). \end{aligned}$$

Then

$$\begin{aligned} &\left\|\frac{R(\mu, B, C) - R(\lambda, B, C)}{\mu - \lambda} - R(\lambda, B, C)CR(\lambda, B, C)\right\| \\ &= \left\|\sum_{k=2}^{\infty}(\mu - \lambda)^{k-1}(R(\lambda, B, C)C)^{k}R(\lambda, B, C)\right\| \\ &\leq |\lambda - \mu| \sup_{k\geq 2}\|(\mu - \lambda)^{k-2}(R(\lambda, B, C)C)^{k}R(\lambda, B, C)\|, \end{aligned}$$

hence

$$\lim_{\mu\to\lambda}\left\|\frac{R(\mu, B, C) - R(\lambda, B, C)}{\mu - \lambda} - R(\lambda, B, C)CR(\lambda, B, C)\right\| = 0.$$

Therefore, (5.2) is satisfied. □

From Theorem 5.7 and $B = I$, we get.

Theorem 5.7. *Let $\mathcal{E}$ be a non-Archimedean Banach space over an algebraically closed field $\mathbb{K}$, let $C \in \mathcal{L}(\mathcal{E})$ such that $R(\lambda, C)$ is analytic on $\rho(C)$. Then*

$$\frac{d}{d\lambda}R(\lambda, C) = R(\lambda, C)^2.$$

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