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Decomposition of modules over generalized Dickson algebras

Sergey V. LUDKOVSKY
MIREA – Russian Technological University
78 Vernadsky Av., Moscow 119454, Russian Federation

Abstract. The article is devoted to modules over generalized Dickson algebras. These algebras are nonassociative and generally can be nonalternative. They compose an important class of algebras and an area in mathematics. Left, right and two-sided modules over generalized Dickson algebras are studied. Their structure and submodules are investigated. Bimodules with involution are scrutinized over generalized Dickson algebras with involution. Such bimodules have specific features caused by involution. Minimal submodules and decomposition of modules are investigated. In particular, cyclic submodules are studied.

Keywords: module, decomposition, generalized Dickson algebra, involution, ring

Mathematics Subject Classification: 15A04, 15A69, 16D70, 17A05, 17A36.

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НАУЧНАЯ СТАТЬЯ

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Разложение модулей над обобщенными алгебрами Диксона

Сергей Викторович ЛЮДКОВСКИЙ
МИРЭА – Российский технологический университет
119454, Российская Федерация, г. Москва, просп. Вернадского, 78

Аннотация. Статья посвящена модулям над обобщенными алгебрами Диксона. Эти алгебры неассоциативны и в общем случае могут быть неальтернативными. Они составляют важный класс алгебр и раздел математики. В работе изучаются левые, правые и двусторонние модули над обобщенными алгебрами Диксона. Исследуется их структура и подмодули. Особое внимание уделено бимодулям с инволюцией над обобщенными алгебрами Диксона с инволюцией. Такие бимодули имеют специфические особенности, вызванные наличием инволюции. Исследуются минимальные подмодули и разложение модулей. В частности, изучаются циклические подмодули.

Ключевые слова: модуль, разложение, обобщенная алгебра Диксона, инволюция, кольцо

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Introduction

Dickson algebras compose a great class of nonassociative algebras (see [1, 2]). They are formed by induction using a doubling procedure of a smashed product (see [3–6] and references therein). This class of algebras is the generalization of the octonion (Cayley) algebra. There are wide-spread applications of Dickson algebras in the theory of Lie groups and algebras (see [7–11]) and their generalizations (see [12]), noncommutative mathematical analysis, noncommutative geometry (see [13, 14]), operator theory (see [15, 16]), PDE (see [17]), elementary particle physics and quantum field theory (see [18]). In the aforementioned areas naturally modules over Dickson algebras are very important, but they are only a little studied.

In this article left, right and two-sided modules over generalized Dickson algebras are studied. They are complicated in comparison with alternative algebras. Specific definitions and notations are given (see Definitions 1.1, 1.2, 1.3, 2.1, 2.2, Remark 1.1), because generalized Dickson algebras are neither associative nor alternative. Structure of modules and submodules over generalized Dickson algebras are investigated. For this purpose auxiliary Lemmas 1.1, 1.2, Corollaries 1.1, 1.2, Examples 1.1 and 2.1 are provided. Dickson algebras posses very important involution property. Therefore bimodules with involution are studied in Section 1. Bimodules with an involution are scrutinized in Theorems 1.1, 1.2, Corollary 1.3. For them necessary and sufficient conditions are elucidated. Identities in them are studied in Proposition 1.1. Subbimodules are investigated in Theorem 1.3 and Corollaries 1.4, 1.5. Relations between left, right and two-sided modules over Dickson algebras are given in Corollary 1.6 and Remark 1.3. Bimodules which are not bimodules with involution also are studied (see Proposition 1.2). Left subbimodules are investigated in Theorem 1.4, Proposition 1.3, Corollary 1.7. In particular, cyclic submodules are studied.

All main results of this paper are obtained for the first time.

1. Modules over generalized Dickson algebras

To avoid misunderstandings we recall necessary definitions and notations in Definition 1.1 and Remark 1.1 (see also [1, 3, 4] and Appendix).

D e f i n i t i o n 1.1. Assume that F is an associative commutative and unital ring. Then over F a unital algebra A is considered, which may be generally nonassociative (relative to multiplication $A \times A \to A$). Assume that A is supplied with a scalar involution $a \mapsto \bar{a}$ so that its norm N and trace T maps have values in F and fulfil conditions:

$$a\bar{a} = N(a)1 \text{ with } N(a) \in F, \tag{1.1}$$

$$a + \bar{a} = T(a)1 \text{ with } T(a) \in F, \tag{1.2}$$

$$T(ab) = T(ba) \tag{1.3}$$

for each a and b in A.

If a scalar $f \in F$ satisfies the condition: $\forall a \in A \ \ fa = 0 \Rightarrow a = 0$, then such element f is called cancelable. Using a cancelable scalar f the Dickson doubling procedure provides new algebra $C(A, f)$ over F such that:

$$C(A, f) = A \oplus A\mathbf{l}, \tag{1.4}$$

$$(a + b\mathbf{l})(c + d\mathbf{l}) = (ac - f\bar{d}b) + (da + b\bar{c})\mathbf{l} \text{ and} \tag{1.5}$$

$$\overline{(a+b\mathbf{l})} = \bar{a} - b\mathbf{l} \tag{1.6}$$

for each a and b in A. Then $\mathbf{l}$ is called a doubling generator.

R e m a r k 1.1. From Definition 1.1 identities follow: $\forall a \in A \ \ \forall b \in A \ \ T(a) = T(a+b\mathbf{l})$ and $N(a+b\mathbf{l}) = N(a) + fN(b)$. The algebra A is embedded into $C(A,f)$ as $A \ni a \mapsto (a,0)$, where $(a,b) = a + b\mathbf{l}$. It is put by induction $A_n(f_{(n)}) = C(A_{n-1}, f_n)$, where $A_0 = A$, $f_1 = f$, $n = 1,2,\dots$, $f_{(n)} = (f_1,\dots,f_n)$. Then $A_n(f_{(n)})$ are generalized Dickson algebras, when F is not a field, or Dickson algebras, when F is a field, where $1 \le n \in \mathbf{N}$.

If the characteristic of F is $char(F) \ne 2$, then the imaginary part of a Dickson number z is defined by:

$$Im(z) = z - T(z)/2,$$

hence $N(a) := N_2(a,\bar{a})/2$, where $N_2(a,b) := T(a\bar{b})$.

If the doubling procedure starts from $A = F1 =: A_0$, then $A_1 = C(A, f_1)$ is a $*$-extension of F.

R e m a r k 1.2. We consider also the following generalizations of the Dickson algebras. Let F be a commutative associative unital ring of characteristic

$$char(F) \ne 2; \tag{1.7}$$

an algebra B has a structure of a F-bimodule with

$$\begin{gathered} x+y = y+x, \quad (x+y)+z = x+(y+z), \\ a(a_1x) = (aa_1)x, \quad (xa_1)a = x(aa_1) \ \text{ and such that } \ ax = xa, \end{gathered} \tag{1.8}$$

for each a and a_1 in F, x, y and z in B, B as the F-bimodule is free and isomorphic with the direct sum

$$B \simeq \bigoplus_{j=0}^{n} F i_j \tag{1.9}$$

with elements $i_j \in B$ for each $j = 0,\dots,n$, satisfying $T_k i_l = \xi_{k,l} i_l$, where $T_k x = (i_k x) i_k$, $\xi_{k,l} \in F$ for each k, l in $\{0,1,2,\dots,n\}$, x in B, where $n > 2$ is a natural number,

$$\xi = (\xi_{k,l})_{k,l=1,\dots,n+1} \tag{1.10}$$

is a $(n+1)\times(n+1)$ matrix having matrix elements $\xi_{k,l}$ such that the corresponding F-linear operator is invertible.

It will frequently be useful also the additional condition

$$i_j i_j = v_j i_0 \tag{1.11}$$

with nonzero cancelable v_j in F possessing an inverse $v_j^{-1} \in F$ for each $j = 0,\dots,n$.

Lemma 1.1. *Let an algebra B satisfy Conditions (1.7)–(1.10) in Remark 1.2. Then there exist F-linear operators $\pi_j : B \to F i_j$ which are F-linear combinations of the operators $T_0,\dots,T_n$ for each $j \in \{0,1,\dots,n\}$ such that $\sum_{j=0}^{n} \pi_j = id_B$, where $id_B(x) = x$ for each $x \in B$.*

P r o o f. From the conditions of this lemma it follows that there exists an inverse operator having matrix $p = \xi^{-1}$ with matrix elements $p_{k,l}$ belonging to F. Then we put $\pi_j(x) = \sum_{k=0}^n p_{j,k}T_k(x)$, consequently, $\pi_j(x) = \sum_{l=0}^n x_l\pi_j(i_l)$, where $x_l \in F$ for each l such that $x = \sum_{l=0}^n x_l i_l$, $x \in B$. Then $\pi_j(i_l) = \sum_{k=0}^n p_{j,k}T_k(i_l)$ hence $\pi_j(i_l) = \sum_{k=0}^n p_{j,k}\xi_{k,l}i_l = \delta_{j,l}i_l$, where $\delta_{j,j} = 1$, $\delta_{j,k} = 0$ for each $k \ne j$. Thus $\pi_j(x) = x_j i_j$. □

Corollary 1.1. *Let the algebra B satisfy conditions (1.7)–(1.11) in Remark 1.2. Then $v_j^{-1}i_j\pi_j(x) = x_j$ for each $x \in B$ and $j = 0, \dots, n$, where $x_l \in F$ for each l such that $x = \sum_{l=0}^n x_l i_l$.*

E x a m p l e 1.1. Assume that F is a (commutative associative) field of characteristic $char(F) \ne 2$, B satisfies conditions (1.7)–(1.10) in Remark 1.2, $\{i_0, i_1, \dots, i_n\}$ is a basis of B over F, $det(\xi) \ne 0$. Then there exists an inverse matrix $p = \xi^{-1}$ with matrix elements $p_{k,l}$ belonging to the field F.

In particular, let us choose $B = A_m(f_{(m)})$ such that $2 \le m \in \mathbf{N}$, where F is the field of characteristic $char(F) \ne 2$, $f_1 = 1, \dots, f_m = 1$, $n = 2^m - 1$, $A_0 = F$ with the trivial involution (i. e. $a = \bar{a}$ for each $a \in A_0$), $i_0 = 1$, where $1 = 1_B$ is the unit element in B (see Remark 1.1). Then $\bar{x} = x_0i_0 - x_1i_1 - \dots - x_ni_n$ for each $x \in B$, where $x_0, \dots, x_n$ denote expansion coefficients belonging to F for x such that $x = x_0i_0 + x_1i_1 + \dots + x_ni_n$. Then $T_0(x) = x$, $T_1(x) = -x_0i_0 - x_1i_1 + x_2i_2 + \dots + x_ni_n$, $\dots$, $T_n(x) = -x_0i_0 + x_1i_1 + \dots + x_{n-1}i_{n-1} - x_ni_n$, since $i_0i_k = i_k$, $i_k^2 = -1$, $i_ki_l = -i_li_k$ and $(i_ki_l)i_k = i_l$ for each $k \ne l$ with $k \ge 1$ and $l \ge 1$. Therefore, $\frac{1}{2-2^m}(T_0 + \dots + T_{2^m-1})(x) = \bar{x}$, consequently, $\pi_0(x) = x_0i_0 = \frac{1}{2}(T_0 + \frac{1}{2-2^m}(T_0 + \dots + T_{2^m-1}))(x)$. Then $\pi_0(\bar{i}_kx) = x_ki_0$ for each $k \ge 1$, hence $\pi_k(x) = (\pi_0(\bar{i}_kx))i_k = x_ki_k$. Thus ξ is the invertible matrix.

Lemma 1.2. *Let $A_n = A_n(f_{(n)})$, $A_0 = A$, $2 \le n \in \mathbf{N}$, where A is the commutative associative unital algebra with the trivial involution over the commutative associative unital ring F of characteristic $char(F) \ne 2$. Let $i_0 = 1_A$, $i_{2^{k-1}} = \mathbf{l}_k$ for each $k = 1, \dots, n$, $i_{j_s} = i_{j_{s-1}}\mathbf{l}_{k_s}$ with $j_1 = 2^{k_1-1}$, $j_s = j_{s-1} + 2^{k_s-1}$ for each $s = 2, \dots, p$, $2 \le p \le n$, $1 \le k_1 < \dots < k_p \le n$, where $\mathbf{l}_p$ denotes the doubling generator $\mathbf{l}$ at the p-th step in Formula (1.5) in Definition 1.1. Then $\{i_j : j = 0, 1, \dots, 2^n-1\}$ is a family of generators of A_n over A_0 satisfying the identities:*

$$i_j(i_ju) = (i_ji_j)u, \quad (ui_j)i_j = u(i_ji_j), \quad i_j(vi_j) = (i_jv)i_j, \quad T(i_j(i_kv)) = 0 \tag{1.12}$$

for each $u \in A_n$, $v = \bar{v} \in A_n$ and $j = 0, 1, \dots, 2^n - 1$, $1 \le k \ne j$.

P r o o f. Since the ring F is commutative and associative, then as it is known the left and right F-module structures can be considered as equivalent: $(pp_1)u = p(p_1u) = p_1(pu) = (p_1p)u$ for each p, p_1 in F, $u \in A_n$, by putting $L_p = R_p$ on A_n for each $p \in F$, where $L_pu = pu$, $R_pu = up$ (see [9, Ch. 2]). The algebra A_0 is unital, hence A_1 is unital, and by induction A_n is unital according to Formulas (1.4), (1.5) in Definition 1.1. The elements f_k in F are cancelable for each k, consequently, the product $f_{k_1} \dots f_{k_p}$ is nonzero for each $1 \le k_1 < \dots < k_p \le n$, $p \ge 2$, since A_n is the unital algebra. For each a_0, a_1 in A_0 by the conditions of this lemma $a_0a_1 = a_1a_0$ and $\overline{a_0a_1} = a_0a_1$.

Using Formulas (1.4), (1.5) in Definition 1.1 by induction we deduce that for each x in A_n there exist elements $x_0, \dots, x_{2^n-1}$ in A_0 such that $x = x_0i_0 + \dots + x_{2^n-1}i_{2^n-1}$. That is, $\{i_j : j = 0, \dots, 2^{n-1}\}$ is the family of generators of A_n over A_0. Therefore,

$$i_j(i_jx) = \sum_{m=0}^{2^n-1} i_j(i_jx_mi_m) = \sum_{m=0}^{2^n-1}(i_j(i_ji_m))x_m \quad \text{and} \quad (xi_j)i_j = \sum_{m=0}^{2^n-1} x_m((i_mi_j)i_j). \tag{1.13}$$

From Formula (1.5) in Definition 1.1 we deduce that

$$\mathbf{l}_k(\mathbf{l}_k(a_{k-1}+d_{k-1}\mathbf{l}_k)) = (\mathbf{l}_k\mathbf{l}_k)(a_{k-1}+d_{k-1}\mathbf{l}_k) \text{ and}$$
$$((a_{k-1}+d_{k-1}\mathbf{l}_k)\mathbf{l}_k)\mathbf{l}_k = (a_{k-1}+d_{k-1}\mathbf{l}_k)(\mathbf{l}_k\mathbf{l}_k) \tag{1.14}$$

for each a_{k-1}, d_{k-1} in A_{k-1}, where $k \ge 1$. Note that $\mathbf{l}_k^2 = -f_k$ for each $k \ge 1$, since A_n is the unital algebra. If $\{\mathbf{l}_{k_1},\dots,\mathbf{l}_{k_p}\}_{q(p)}$ denotes an ordered product, where $q(p)$ is a vector indicating an order of pairwise multiplication with a corresponding order of brackets in $\mathbf{l}_{k_1},\dots,\mathbf{l}_{k_p}$, where $1 \le k_l \le n$, $k_l \ne k_s$ for each $l \ne s$, l and s in $\{1,\dots,n\}$, $2 \le p \le n$, then there exist unique $\eta(k_1,\dots,k_p,q(p)) \in \{1,2\}$ and $j(k_1,\dots,k_p,q(p)) \in \{1,\dots,2^n-1\}$ such that $(-1)^\eta\{\mathbf{l}_{k_1},\dots,\mathbf{l}_{k_p}\}_{q(p)} = i_j$, where $\eta = \eta(k_1,\dots,k_p,q(p))$, $j = j(k_1,\dots,k_p,q(p))$. Therefore, from (1.14) by induction in $k = 1,\dots,n$ it follows that $i_j(i_ji_m) = (i_ji_j)i_m$ and $(i_mi_j)i_j = i_m(i_ji_j)$ for each m and j. For each $v = \bar{v} \in A_n$ from Formulas (1.5) and (1.6) in Definition 1.1 it follows that $T(\mathbf{l}_kv) = 0$, $\mathbf{l}_k(v\mathbf{l}_k) = (\mathbf{l}_kv)\mathbf{l}_k = \overline{\mathbf{l}_k(v\mathbf{l}_k)}$ for each $k \ge 1$ and by induction $i_j(vi_j) = (i_jv)i_j$ for each j, since $i_0 = 1$. The latter and (1.13) imply (1.12). □

Corollary 1.2. *If the conditions of Lemma 1.2 are satisfied and F is a field, $A_0 = F$, then $\{i_j : j = 0,\dots,2^n-1\}$ is a basis in A_n (as in the F-linear space).*

D e f i n i t i o n 1.2. Let F be the commutative associative unital ring. Let B be a unital algebra over F with $F \subseteq \mathsf{C}_B(B)$. Let M be a unital left $\mathsf{C}_B(B)$-module:

$$b(b_1u) = (bb_1)u,\ \ b(u+v) = bu+bv,\ \ (b+b_1)u = bu+b_1u,\ \ u+(u_1+v) = (u+u_1)+v \tag{1.15}$$

for each u, u_1, v in M, b, b_1 in $\mathsf{C}_B(B)$. Let μ_1 be a $\mathsf{C}_B(B)$-bilinear map $\mu_1 : B\times M \to M$, that is,

$$\mu_1(x,u+v) = \mu_1(x,u)+\mu_1(x,v),\ \ \mu_1(x+y,u) = \mu_1(x,u)+\mu_1(y,u),\ \ \mu_1(bx,u) = b\mu_1(x,u) \tag{1.16}$$

such that μ_1 is compatible with the left $\mathsf{C}_B(B)$-module structure of M:

$$\mu_1(x,bu) = b\mu_1(x,u) \tag{1.17}$$

for each x, y in B, u, v in M, b in $\mathsf{C}_B(B)$. Then M will be called a left B-module. Shortly $\mu_1(x,u)$ can also be denoted by xu. Similarly is defined a right B-module, or a B-bimodule.

For $B = A_n(f_{(n)})$ with $A_0 = A$, $n \ge 2$, where A is the commutative associative unital algebra with the trivial involution over the associative commutative unital ring F of characteristic $char(F) \ne 2$, if M satisfies conditions (1.15)–(1.17) and

$$i_j(i_jx) = (i_ji_j)x \tag{1.18}$$

for each $x \in M$, $j = 0,\dots,2^{n-1}$, then M will be called a left $A_n(f_{(n)})$-module. Symmetrically is defined the right $A_n(f_{(n)})$-module with condition (1.19) instead of (1.18):

$$(xi_j)i_j = x(i_ji_j) \tag{1.19}$$

for each $x \in M$, $j = 0,\dots,2^{n-1}$. If M satisfies (1.15)–(1.19) and (1.20):

$$L_b = R_b \text{ on } M \text{ for each } b \in \mathsf{C}_B(B), \tag{1.20}$$

then it will be called $A_n(f_{(n)})$-bimodule and denoted by ${}_BM_B$ or shortly by M, where $B = A_n(f_{(n)})$.

If in the B-bimodule M there exists a $\mathsf{C}_B(B)$-linear map $J : M \to M$ such that

$$J(bx) = (Jx)\bar{b} \text{ for each } x \in M \text{ and } b \in B,\ J^2 = I \tag{1.21}$$

$$\text{and } \big((\exists x \in M\ \ \forall b \in B\ \ bx + J(bx) = 0) \Rightarrow (x = 0)\big) \tag{1.22}$$

$$\text{and } (i_j y)i_j = i_j(y i_j) \text{ and } i_j(i_k y) + \overline{i_j(i_k y)} = 0 \tag{1.23}$$

for each $y = Jy$ in M and $j \geq 0$, $j \neq k \geq 1$, where $B = A_n(f_{(n)})$, $I : M \to M$ is the identity map $I = id_M$ on M, $Ix = x$, then M will be called the B-bimodule with the involution J and denoted by ${}_B\hat{M}_B$ or shortly by M. Briefly Jx will also be denoted by $\bar{x}$.

Theorem 1.1. *Let M be the unital $A_n(f_{(n)})$-bimodule with the involution J, let the subalgebra A_0 over F be commutative associative and with the trivial involution $\bar{a} = a$ for each $a \in A_0$, let also $char(F) \neq 2$ and f_j possess an inverse element f_j^{-1} in F relative to multiplication for each $j = 1, \ldots, n$, where $2 \leq n \in \mathbf{N}$. Then there exists an A_0-subbimodule M_0 such that ${}_{A_0}M_{A_0} = \bigoplus_{j=0}^{2^n-1} i_j M_0$ with $JM_0 = M_0$, and $M_0 = \mathsf{C}_M(A_n(f_{(n)}))$, and there exists an A_0-linear map $\hat{\pi}_k$ from M onto $i_k M_0$ with $\hat{\pi}_k \circ \hat{\pi}_j = \delta_{k,j}\hat{\pi}_k$, where $\delta_{k,k} = 1$, $\delta_{k,j} = 0$ if $k \neq j$, for each k and j in $\{0, 1, \ldots, 2^n - 1\}$.*

P r o o f. By virtue of Lemma 1.2 the Dickson algebra $B = A_n(f_{(n)})$ has the family of generators $\beta_n := \{i_j : j = 0, \ldots, 2^n-1\}$ over A_0. By the conditions imposed above in Definition 1.2 the algebra A_0 and the ring F are unital such that there is the natural embedding of F into A_0 as $F1_{A_0}$ and hence into B. Therefore, B contains the Dickson subalgebra $alg_F\beta_n$ over F with generators $i_0, \ldots, i_{2^n-1}$.

Note that $F \subseteq \mathsf{C}_B(B)$ and $Fi_k\beta_n = F\beta_n$, $F\beta_n i_k = F\beta_n$ for each k, since $F \subseteq \mathsf{C}_{A_0}(A_0)$. As in Remark 1.2 let $T_k u = (i_k u)i_k$ for each $u \in B$, and $\hat{T}_k x = (i_k x)i_k$ for each $x \in M$, $k = 0, \ldots, 2^n - 1$. We put $\pi_0(u) = \frac{u+\bar{u}}{2}$ for each $u \in B$, and $\hat{\pi}_0(x) = \frac{x+\bar{x}}{2}$ for each $x \in M$. Let $M_0 = \hat{\pi}_0(M)$, hence $M_0 = \bigcup\{y \in M : \exists x \in M, y = \frac{x+\bar{x}}{2}\}$. On the other hand, $R_b x = L_b x$ for each $b \in \mathsf{C}_B(B)$ and $x \in M$ by Definition 1.2, where $L_b u = bu$, $R_b u = ub$ for each u, b in B. The algebra A_0 is commutative and associative with $a_0 = \bar{a}_0$ for each $a_0 \in A_0$, consequently, $A_0 \subseteq \mathsf{C}_B(B)$ and hence $\overline{a_0 y} = \bar{y}\bar{a}_0 = y a_0 = a_0 y$ for each $y \in M_0$ and $a_0 \in A_0$. Therefore, M_0 is the A_0-subbimodule in M, since $A_0 \subset B$ and M has also the structure of the A_0-bimodule ${}_{A_0}M_{A_0}$. It follows that $y = \bar{y}$ and $\hat{\pi}_0(y) = y$ for each $y \in M_0$, hence $\hat{\pi}_0 \circ \hat{\pi}_0 = \hat{\pi}_0$, where as usually $(g \circ h)(v) = g(h(v))$ denotes the composition of maps g and h with a variable v of h.

For each $x \in M$ there is the decomposition $x = y + z$ such that $y = \frac{x+\bar{x}}{2}$, $z = \frac{x-\bar{x}}{2}$. Let $M_- := \{z \in M : \bar{z} = -z\}$. Evidently, $M_0 \cap M_- = \{0\}$ and M_- is the A_0-subbimodule in ${}_{A_0}M_{A_0}$ such that $M_0 \oplus M_- = {}_{A_0}M_{A_0}$, since $J : M \to M$ and $J^2 = I$, also $B_0 \cap B_- = \{0\}$, $B_0 = \pi_0(B)$, $B_0 = A_0$, $B_- := \{u \in B : \exists v \in B, u = \frac{v-\bar{v}}{2}\}$.

We put

$$S_n(f_{(n)}) := \bigcup \Big\{ f \in F : \exists p \in \{1, \ldots, n\}\ \exists j_1 \in \{1, \ldots, n\}\ \ldots\ \exists j_p \in \{1, \ldots, n\}$$
$$\exists \alpha_1 \in \mathbf{Z}\ \ldots\ \exists \alpha_p \in \mathbf{Z}\ \exists v \in \{-1, 1\} : f = v f_{j_1}^{\alpha_1} \ldots f_{j_p}^{\alpha_p} \Big\},$$

consequently, $i_k^2 = s_k \in S$ for each $k \geq 1$, where $S = S_n(f_{(n)})$. Note that if $y \in M_0$, $z \in M_-$, $b \in B_-$, then $(by - yb) \in M_0$, $(by + yb) \in M_-$, $bz + zb \in M_0$, $bz - zb \in M_-$.

On the other hand, $(i_k y = y i_k) \Leftrightarrow ((i_k y) i_k = y s_k) \Leftrightarrow (i_k(y i_k) = y s_k)$ for each $k \ge 1$, consequently, from conditions (1.21), (1.22), (1.23) in Definition 1.2 it follows that $i_k y = y i_k$ for each $k = 0, \dots, 2^n - 1$, since $s_k \in S \subset F$, $i_0 = 1$. This implies that $i_k y \in M_-$ for each $k \ge 1$. By the A_0-linearity and Lemma 1.2 this implies that $M_0 \subseteq Com_M(B)$.

Then we put $\hat{\pi}_j(x) = -s_j^{-1}(\hat{\pi}_0(\bar{b}_j x)) i_j$ for each $x \in M$, $j = 1, \dots, 2^n - 1$. Notice that $\hat{\pi}_k(x) = \hat{\pi}_k(y) + \hat{\pi}_k(z) = \hat{\pi}_k(z)$ for each $k \ge 1$ and $x \in M$, where $y \in M_0$, $z \in M_-$, $x = y + z$, $y = \frac{I+J}{2} x$, $z = \frac{I-J}{2} x$, since $i_k y \in M_-$ and $\hat{\pi}_0(M_-) = \{0\}$. Thus $\hat{\pi}_k \circ \hat{\pi}_0 = 0$ for each $k \ge 1$, since $\hat{\pi}_k(M_0) = \{0\}$ and $\hat{\pi}_0(M) = M_0$. From $i_k^2 x = i_k(i_k x) = s_k x$ with $s_k \in S$ for each $k \ge 1$, $char(F) \ne 2$, $F \subseteq A_0 \subseteq \mathsf{C}_B(B)$ and the conditions (1.15)–(1.19) in Definition 1.2 it follows that $L_{i_k} : M \to M$ and similarly $R_{i_k} : M \to M$ are A_0-linear bijections for each $k = 0, \dots, 2^n - 1$, since M is the unital B-bimodule with involution, since the algebra B is unital, $i_0 = 1_B$. Then we deduce that $\hat{\pi}_k(x) = \frac{-1}{2 s_k}([I + J](\bar{b}_k x)) i_k = \frac{-1}{2}[s_k^{-1}(\bar{b}_k x) i_k + \bar{x}]$ for each $x \in M$ and $k = 1, \dots, 2^n - 1$. Therefore, $\hat{\pi}_k : M_- \to M_-$ for each $k \ge 1$, since $z = -\bar{z}$ for each $z \in M_-$. This implies that $\hat{\pi}_0 \circ \hat{\pi}_k(x) = \hat{\pi}_0(\hat{\pi}_k(z)) = 0$ for each $x \in M$ and $k \ge 1$, since $M_0 \cap M_- = \{0\}$, where $z = \frac{I-J}{2} x$. Then we infer that $\hat{\pi}_k \circ \hat{\pi}_k(x) = -\hat{\pi}_k(\bar{x}) = \hat{\pi}_k(x)$ for each $x \in M$ and $k \ge 1$, since $\hat{\pi}_k(x) = \hat{\pi}_k(z)$ with $z = \frac{I-J}{2} x$, since $L_{i_k} R_{i_k} \hat{\pi}_0 = L_{i_k^2} \hat{\pi}_0$, since $\hat{\pi}_0(M) = M_0$. Particularly $\hat{\pi}_k(i_k y) = i_k y$ for each $k \ge 1$ and $y \in M_0$, since $i_k y = y i_k$. This implies that $\hat{\pi}_k(M) = \hat{\pi}_k \circ \hat{\pi}_k(M) = \hat{\pi}_k(i_k M_0) = i_k M_0$, $i_k M_0 = M_0 i_k$ for each $k \ge 0$, since $i_0 = 1_B$. Note that $i_k M_0 \subset M_-$ and $\hat{\pi}_0(\bar{b}_k x) = \hat{\pi}_0(\bar{b}_k z)$ with $z = \frac{I-J}{2} x$ for each $k \ge 1$ and $x \in M$. Thus $\hat{\pi}_0|_{M_0} = id_{M_0}$, $\hat{\pi}_0|_{M_-} = 0$, where $id_{M_0}(y) = y$ for each $y \in M_0$. Therefore, $\hat{\pi}_j \circ \hat{\pi}_k = 0$ for each $j \ne k$, since $i_j M_0 \cap i_k M_0 = \{0\}$, since $i_j(i_k y) + \overline{i_j(i_k y)} = 0$ for each $y = Jy$ in M and $j \ge 0$, $j \ne k \ge 1$, since f_j is invertible relative to multiplication in F for each j.

Then we put $\hat{K} = \sum_{j=0}^{2^n-1} \hat{\pi}_j$ on M, and $K = \sum_{j=0}^{2^n-1} \pi_j$ on B. These operators are idempotent $\hat{K}^2 = \hat{K}$ and $K^2 = K$, since $\hat{\pi}_j \circ \hat{\pi}_k = \delta_{j,k} \hat{\pi}_j$ and $\pi_j \circ \pi_k = \delta_{j,k} \pi_j$ for each $j, k = 0, \dots, 2^n - 1$. Hence $I - \hat{K}$ also is the idempotent operator.

It is known that the minimal subalgebra $A_{(j,k)}$ in $A_n(f_{(n)})$ generated by $\{A_0, i_j, i_k\}$ is associative for each $j, k = 0, \dots, 2^n - 1$, since F and A_0 are commutative and associative by the conditions of this theorem (see [1,4,9]). Therefore, $M_{(j,k)} := M_0 \oplus i_j M_0 \oplus i_k M_0 \oplus (i_j i_k) M_0$ is the $A_{(j,k)}$-subbimodule with involution in M, since $i_k M_0 = M_0 i_k$ for each k, $i_j M_0 \cap i_k M_0 = \{0\}$ for each $j \ne k$, $i_j i_k \in G$, where $G = G_n(f_{(n)}) = \{i_0, \dots, i_{2^n-1}\} \cdot S$, $F \subseteq A_0$.

On the other hand, $\hat{K} y = \hat{\pi}_0 y = y$ and $\hat{K}(i_j y) = \hat{\pi}_j(i_j y) = i_j y$ for each $y \in M_0$ and $j \ge 1$, since $\hat{\pi}_k(i_j y) = \hat{\pi}_k \circ \hat{\pi}_j(i_j y) = 0$ for each $j \ne k$. Then we deduce that $\hat{K} M_- = \bigoplus_{j=1}^{2^n-1} i_j M_0$, since $L_{i_j} : M_0 \to M_-$ and $\hat{\pi}_j M_- = i_j M_0$ for each $j \ge 1$, since $(\hat{\pi}_j M_-) i_j = M_0$, $M_0 i_j = i_j M_0$. Hence $\hat{\pi}_k(i_k P) = \hat{\pi}_0(P) = \{0\}$, where $P := M \ominus (\bigoplus_{j=0}^{2^n-1} i_j M_0)$. Notice that P is the proper $A_n(f_{(n)})$-subbimodule with involution in M, that is P satisfies conditions (1.18)–(1.23) in Definition 1.2. On the other side, the condition

$$\big((\exists x \in M \ \forall b \in B \ bx + J(bx) = 0) \Rightarrow (x = 0)\big) \text{ is equivalent to}$$
$$\big((\exists x \in M \ \forall j \in \{0, \dots, 2^n - 1\} \ i_j x + J(i_j x) = 0) \Rightarrow (x = 0)\big),$$

since for each $b \in B$ there exist $a_0, \dots, a_{2^n-1}$ in A_0 such that $b = a_0 i_0 + \dots + a_{2^n-1} i_{2^n-1}$ by Lemma 1.2. From $\hat{\pi}_0(P) = \{0\}$ and $P_0 = \hat{\pi}_0(P)$ it follows that $P_0 = \{0\}$, consequently, $P_- = \{0\}$, since $P_0 = i_j \hat{\pi}_j(P_-) = \{0\}$ for each $j \ge 1$. Thus $P = \{0\}$, consequently, $\hat{K} = I$ on M and hence ${}_{A_0}M_{A_0} = \bigoplus_{j=0}^{2^n-1} i_j M_0$ and consequently, $M_0 = Com_M(B)$, where M is considered as the A_0-bimodule ${}_{A_0}M_{A_0}$, since $M_- \cap Com_M(B) = \{0\}$. Analogously ${}_{A_0}B_{A_0} = \bigoplus_{j=0}^{2^n-1} i_j B_0$ and $B_0 = Com_B(B)$, $B_0 = \pi_0(B)$, where B is considered as the A_0-bimodule ${}_{A_0}B_{A_0}$.

Therefore, for each $y \in M_0$ and for each $j \neq k$ such that $j \geq 1$ and $k \geq 1$ we infer that $\hat{\pi}_t((i_j y)i_k) = 0$ and $\hat{\pi}_t(i_j(yi_k)) = 0$ for each $t \neq r$, $\hat{\pi}_r((i_j y)i_k) = (i_j y)i_k$ and $\hat{\pi}_r(i_j(yi_k)) = i_j(yi_k)$, where $r \in \{1, \dots, 2^n - 1\}$ is such that $i_j i_k \in i_r S$, since $(i_j y)i_k \in M_{(j,k)-}$, $i_j(yi_k) \in M_{(j,k)-}$. We put $v = i_j(yi_k) - (i_j y)i_k$. Therefore, $v \in L_{i_r} M_0 \subset M_{(j,k)-}$. From Formulas (1.21), (1.22) and (1.23) in Definition 1.2 we deduce that $v = \bar{v}$. Thus $v \in M_0 \cap M_- = \{0\}$, that is $v = 0$. Hence $(i_j y)i_k = i_j(yi_k)$ for each $j \neq k$ such that $j \geq 1$ and $k \geq 1$. For $j = 0$ or $k = 0$ evidently $i_j(yi_k) = (i_j y)i_k$, since $i_0 = 1_B$. Using $M_0 = Com_M(B)$ and Conditions (1.21), (1.22), (1.23) in Definition 1.2 we infer that $i_j(yi_k) = -(yi_k)i_j$ for each $y \in M_0$ and $j \neq k$ such that $j \geq 1$, $k \geq 1$. Then it is similarly deduced that $i_j(i_k y) = (i_j i_k)y$ and $(yi_k)i_j = y(i_k i_j)$ for each $j \neq k$ in $\{1, \dots, 2^n - 1\}$, $y \in M_0$, since $v + v_1 = 0$ and $\bar{v} = v_1$ with $v = i_j(i_k y) - (i_j i_k)y$, $v_1 = (yi_k)i_j - y(i_k i_j)$, since $v \in M_-$, $v_1 \in M_-$, $M_0 \cap M_- = \{0\}$, since i_j, i_k, $i_j i_k$ belong to B_-. If $j = 0$ or $k = 0$, evidently $i_j(i_k y) = (i_j i_k)y$ and $(yi_k)i_j = y(i_k i_j)$ for each $y \in M_0$. By the A_0-linearity and Lemma 1.2 we infer that $M_0 \subseteq N_M(B)$, consequently, $\mathsf{C}_M(B) = M_0$. □

Corollary 1.3. *Let the conditions of Theorem 1.1 be satisfied and $n = 3$. Then $b(bx) = (bb)x$, $(bx)b = b(xb)$, $(xb)b = x(bb)$ for each $x \in M$ and $b \in B$.*

P r o p o s i t i o n 1.1. *If the conditions of Theorem 1.1 are satisfied and there is some equality with a finite sum like*

$$\sum_{\theta \in \mathbf{S}_m;\, k_1, \dots, k_m;\, l} \gamma_{\theta;k_1,\dots,k_m;l} \{d_{\theta(k_1)} \dots d_{\theta(k_m)}\}_{q_{l,\theta}(m)} = 0$$

in $A_n(f_{(n)})$, where $d_{k_j} \in A_n(f_{(n)})$, $\gamma_{\theta;k_1,\dots,k_m;l} \in A_0$ for each k_j, j, l, θ, then there exists a corresponding identity in M.

P r o o f. For the identity satisfying the conditions of this proposition we use the decomposition ${}_{A_0}M_{A_0} = \bigoplus_{j=0}^{2^n-1} i_j M_0$ and $JM_0 = M_0$, where $M_0 = \mathsf{C}_M(B)$. Then we substitute one of d_{k_j} on $d_{k_j} y$ with an arbitrary fixed nonzero $y \in M_0$ for each additive $\{d_{\theta(k_1)} \dots d_{\theta(k_m)}\}_{q_{l,\theta}(m)}$, where $\mathbf{S}_m$ denotes the symmetric group of $\{1, \dots, m\}$, $\theta : \{1, \dots, m\} \to \{1, \dots, m\}$ is a bijection for each $\theta \in \mathbf{S}_m$, $q_{l,\theta}(m)$ is a vector indicating an order of pairwise multiplications in $\{\dots\}$. Then it is possible to make sums of such type equalities with multipliers from A_0. □

This proposition shows that definitions above are natural, because particularly the algebra has also the structure of the module over itself. There may other equivalent definitions be given.

Theorem 1.2. *Assume that F is a commutative associative unital ring, $char(F) \neq 2$, a unital algebra A_0 over F is associative and commutative with the trivial involution $\bar{a} = a$ for each $a \in A_0$, M_0 is a unital A_0-bimodule, $B = A_n(f_{(n)})$ is the generalized Dickson algebra, and f_j possess an inverse element f_j^{-1} in F relative to multiplication for each $j = 1, \dots, n$, ${}_{A_0}M_{A_0} = \bigoplus_{j=0}^{2^n-1} i_j M_0$ such that $M_0 = \mathsf{C}_M(B)$, where $n \geq 2$. Then ${}_{A_0}M_{A_0}$ can be supplied with B-bimodule with involution ${}_B\hat{M}_B$ structure.*

P r o o f. We put $by = yb$, $a(by) = (ab)y$, $a(yb) = (ay)b$, $(ya)b = y(ab)$, $J(by) = \bar{b}y$ for each $y \in M_0$, a and b in B, $J(x + z) = Jx + Jz$ for each x and z in $M = \bigoplus_{j=0}^{2^n-1} i_j M_0$, since $M_0 = \mathsf{C}_M(B)$ and $i_0 y = y$ for each $y \in M_0$. Therefore, $Jx = x_0\bar{b}_0 + \ldots + x_{2^n-1}\bar{b}_{2^n-1}$ for each $x = x_0 i_0 + \ldots + x_{2^n-1} i_{2^n-1}$ in M with $x_0, \ldots, x_{2^n-1}$ in M_0, consequently, $J^2 = I$ and hence J is the involution on M. In view of Lemma 1.2 the equalities $by = yb$, $a(by) = (ab)y$, $a(yb) = (ay)b$, $(ya)b = y(ab)$ for each $y \in M_0$, a and b in B, supply M with properties (1.15)–(1.23) in Definition 1.2, since the minimal subalgebra $A_{(j,k,l)}$ in $A_n(f_{(n)})$ generated by $\{A_0, i_j, i_k, i_l\}$ is alternative for each j, k, l in $\{0, \ldots, 2^n - 1\}$ (see [1, 4, 9]), since F and A_0 are unital, associative and commutative, $\bar{a} = a$ for each $a \in A_0$, since each $x \in M$ has the decomposition $x = x_0 i_0 +, \ldots, +x_{2^n-1} i_{2^n-1}$ in M with $x_0, \ldots, x_{2^n-1}$ in M_0. □

D e f i n i t i o n 1.3. If M is the left B-module (see Definition 1.2), E is a subset in B, D is a subset in B (or in M), then

$$E \cdot D = \bigcup \{ed : e \in E,\ d \in D\},$$

$$ED = \bigcup \{x = \sum_{k=1}^{m} e_k d_k : m \in \mathbf{N},\ \forall k\ e_k \in E,\ d_k \in D\}$$

denote subsets in B (or in M correspondingly). Then it is put

$$E^{(1)} = E,\quad (E \cdot D)^{(1)} = E \cdot D;$$

$$\forall n > 1\ \ dE^{(n)} = E \cdot E^{n-1},\quad (E \cdot D)^{(n)} = E \cdot (E \cdot D)^{(n-1)};$$

$$E^{<1>} = E,\quad (ED)^{<1>} = ED,$$

$$\forall n > 1\ \ E^{<n>} = EE^{n-1},\quad (ED)^{<n>} = E(ED)^{<n-1>};$$

$$E^{(\infty)} = \bigcup_{n=1}^{\infty} E^{(n)},\quad d(E \cdot D)^{(\infty)} = \bigcup_{n=1}^{\infty} (E \cdot D)^{(n)};$$

$$E^{<\infty>} = \sum_{n=1}^{\infty} E^{<n>},\quad (ED)^{<\infty>} = \sum_{n=1}^{\infty} (ED)^{<n>}.$$

If N is a left B-submodule in M such that $\exists d \in M$, $D = \{d\}$, $E = B$, $N = (B\{d\})^{<\infty>}$, then N is called a cyclic left B-submodule in M generated by d.

Similar notations are for right B-modules or B-bimodules.

If M is the B-bimodule, then

$$(E \cdot D)^{(1,1)} = (E \cdot D) \cup (D \cdot E),\quad (ED)^{<1,1>} = (ED) + (DE),$$

$$\forall n > 1\ \ (E \cdot D)^{(n,n)} = (E \cdot (E \cdot D)^{(n-1,n-1)}) \cup ((E \cdot D)^{(n-1,n-1)} \cdot E),$$

$$(ED)^{<n,n>} = (E(ED)^{<n-1,n-1>}) + ((ED)^{<n-1,n-1>} E);$$

$$(E \cdot D)^{(\infty,\infty)} = \bigcup_{n=1}^{\infty} (E \cdot D)^{(n,n)},\quad (ED)^{<\infty,\infty>} = \sum_{n=1}^{\infty} (ED)^{<n,n>}.$$

If N is a B-subbimodule in M such that $\exists d \in M$, $D = \{d\}$, $E = B$, $N = (B\{d\})^{<\infty,\infty>}$, then N is called a cyclic B-subbimodule in M generated by d.

If F is the field and V is an $A_n(f_{(n)})$-subbimodule with the involution in an $A_n(f_{(n)})$-bimodule with the involution, then $dim_F V$ denotes the dimension of V over F.

Theorem 1.3. *Let the conditions of Theorem 1.1 be satisfied, $D \subset M$. Then $(BD)^{<m,m>} \subseteq (BD)^{<m+1,m+1>}$ for each $m \geq 1$, $(BD)^{<k,k>} = (BD)^{<4,4>}$ for each $k \geq 4$. Moreover, $(BD)^{<4,4>}$ is the B-subbimodule with involution in M and $(BD)^{<4,4>} = (BD)^{<\infty,\infty>}$, $(BD)^{<\infty,\infty>} = (B\bar{D})^{<\infty,\infty>}$.*

P r o o f. The algebra $B = A_n(f_{(n)})$ is unital, the B-bimodule with involution M is unital, $n \ge 2$, by the imposed conditions. Therefore, $(BD)^{<m,m>} \subseteq (BD)^{<m+1,m+1>}$ for each $m \ge 1$.

For each $x \in D$ the element $\hat{\pi}_j(x)$ belongs to $(B(B\{x\}))B$ for each $j = 0, \dots, 2^n - 1$, since

$$\bar{x} = \frac{1}{2-2^n}\big(x + \frac{(\bar{b}_1 x)i_1}{s_1} + \dots + \frac{(\bar{b}_{2^n-1}x)i_{2^n-1}}{s_{2^n-1}}\big)$$

by Theorem 1.1, where s_j is invertible in F (relative to multiplication) for each $j \ge 1$. Evidently, $(B(B\{x\}))B \subseteq (BD)^{<3,3>}$ for each $x \in D$. On the other hand, $R_{i_k}\hat{\pi}_j(x)$ belong to $(BD)^{<4,4>}$ for each j and k in $\{0, \dots, 2^n - 1\}$, $x \in D$, since by Theorem 1.1 $\hat{\pi}_j(x) \in i_j M_0$, each x in M has the decomposition $x = x_0 i_0 + \dots + x_{2^n-1} i_{2^n-1}$ with $x_0, \dots, x_{2^n-1}$ in M_0, $\hat{\pi}_j(x) = x_j i_j$, $M_0 = \mathsf{C}_M(B)$. This implies that $(BD)^{<4,4>} = \sum_{k=0}^{2^n-1} i_k V_0$ with $V_0 = span_F \bigcup \big\{x_j \,:\, \exists x \in D \ \exists j \in \{0, \dots, 2^n - 1\} \ x_j = \frac{i_j}{s_j}\hat{\pi}_j(x)\big\}$, where $span_F Q$ denotes the F-linear span of a subset Q in M. Certainly, $V_0 \subset M_0$ and consequently, $(BD)^{<4,4>}$ is the B-subbimodule with involution in M and $(BD)^{<4,4>} = (BD)^{<\infty,\infty>}$, $(BD)^{<\infty,\infty>} = (B\bar{D})^{<\infty,\infty>}$. $\square$

Corollary 1.4. *If the conditions of Theorem 1.3 are satisfied and F is the field, then $dim_F (BD)^{<\infty,\infty>} = 2^n dim_F V_0$ and $dim_F V_0 \le 2^n card(D)$.*

Corollary 1.5. *Let F be a commutative associative unital ring, $char(F) \ne 2$, let A_0 be a commutative associative unital algebra over F with trivial involution $a = \bar{a}$ for each $a \in A_0$, $2 \le n \in \mathbf{N}$, f_j be invertible in F relative to multiplication for each $j = 1, \dots, n+1$. Let also N be an $A_n(f_{(n)})$-bimodule with involution and N be contained in some $A_{n+1}(f_{(n+1)})$-bimodule P such that $\mathsf{C}_N(A_n(f_{(n)})) = \mathsf{C}_N(A_{n+1}(f_{(n+1)}))$, then $M = N \oplus (N\mathbf{l}_{n+1})$ is an $A_{n+1}(f_{(n+1)})$-bimodule with involution and $M_0 = N_0$.*

P r o o f. By virtue of Theorem 1.1 N has the decomposition $N = \bigoplus_{k=0}^{2^n-1} N_0 i_k$ with $N_0 = \mathsf{C}_N(A_n(f_{(n)}))$, hence

$$M = \bigoplus_{j=0}^{2^{n+1}-1} N_0 i_j.$$

From Theorem 1.2 it follows that M is the $A_{n+1}(f_{(n+1)})$-bimodule with involution and $M_0 = N_0$, since $\mathsf{C}_N(A_n(f_{(n)})) = \mathsf{C}_N(A_{n+1}(f_{(n+1)}))$. $\square$

R e m a r k 1.3. For the generalized Dickson algebra $B = A_n(f_{(n)})$ with $n \ge 2$, there is its unvolutorial algebra $\bar{B}$, which as an F-linear space, is the same, but has the multiplication obtained from B by the following formula: $\bar{a} \diamond \bar{b} = \bar{c}$ with $\bar{c} = ba$ induced from B by the involution operator $Jb = \bar{b}$ for each $\bar{a}$, $\bar{b}$ in $\bar{B}$, an addition in $\bar{B}$ is induced by that of in B.

Therefore, the left $\bar{B}$-module ${}_{\bar{B}}M$ also has the structure of the right B-module M_B such that $\bar{a} \diamond (\bar{b} \diamond x) = (xb)a$, where $\diamond$ denotes the multiplication of x in M on $\bar{b}$, $\bar{a}$ in $\bar{B}$. Using the tensor product over F and the involutorial algebra $\bar{B}$ instead of the opposite algebra B^{op} one gets the involutorial enveloping algebra $\check{B}^e = B \bigotimes_F \bar{B}$ instead of the enveloping algebra $B^e = B \bigotimes_F B^{op}$. Then the left $\check{B}^e$-module ${}_{\check{B}^e}M$ also has the structure of B-bimodule ${}_B M_B$, but generally it may not have the structure of the B-bimodule with involution ${}_B\hat{M}_B$.

P r o p o s i t i o n 1.2. *Let $B = A_n(f_{(n)})$, $n \ge 2$, where A_0 is the commutative associative unital algebra with trivial involution $\bar{a} = a$ for each $a \in A_0$ over the commutative associative unital ring F, $char(F) \ne 2$, f_j is invertible in F relative to multiplication for each $j = 1, \dots, n$. Then there exist B-bimodules which are not B-bimodules with involution.*

P r o o f. Take $A_{n+p}(f_{(n+p)})$ with $n \ge 2$ and $p \ge 1$, with f_j invertible in F for each $j = n+1, \dots, n+p$. Then $M = A_{n+p}(f_{(n+p)})$ has the structure of the B-bimodule ${}_BM_B$, but it is not the B-bimodule with involution by Theorems 1.1 and 1.2. That is, this M does not satisfy conditions (1.21), (1.22), (1.23) in Definition 1.2. □

Theorem 1.4. *Let ${}_BN$ the left B-module with $B = A_n(f_{(n)})$, $n \ge 2$, where A_0 is the commutative associative unital algebra with trivial involution $\bar{a} = a$ for each $a \in A_0$ over the commutative associative unital ring F, $char(F) \ne 2$, f_j is invertible in F relative to multiplication for each $j = 1, \dots, n$. Let $D \subset N$, $N \subseteq M$, where M has the structure of the B-bimodule with involution ${}_B\hat{M}_B$. Then $(BD)^{<m>} = (BD)^{<1>}$ for each $1 < m \le \infty$ and $(BD)^{<1>}$ is the left B-submodule in ${}_BN$.*

P r o o f. In view of Lemma 1.2 $(BD)^{<1>}$ is the A_0-linear span $span_{A_0}Q$ of the family $Q = \{i_jx : x \in D, j \in \{0, \dots, 2^n - 1\}\}$. By virtue of Theorem 1.1 each element x in M has the decomposition $x = x_0i_0 + \dots + x_{2^n-1}i_{2^n-1}$ with $x_0, \dots, x_{2^n-1}$ belonging to M_0, that is $x = \beta[x]$, where $\beta = \beta_n = (i_0, \dots, i_{2^n-1})$, $[x]^t = (x_0, \dots, x_{2^n-1})$, U^t denotes a transposed matrix of a matrix U. Consequently, $i_jx = x_0(i_ji_0) +, \dots, +x_{2^n-1}(i_ji_{2^n-1})$ for each $j \in \{0, \dots, 2^n - 1\}$, since $M_0 = \mathsf{C}_M(B)$.

On the other hand, $\{i_{j_1} \dots i_{j_m}\}_{q(m)} \in G$ for each $j_1, \dots, j_m$ in $\{0, \dots, 2^n - 1\}$, $2 \le m \in \mathbf{N}$, where $G = G_n(f_{(n)})$, where $q(m)$ is a vector indicating an order of pairwise multiplications in $\{\dots\}$. Note that $si_jG = G$ for each $j \in \{0, \dots, 2^n - 1\}$ and $s \in S$, where $S = S_n(f_{(n)})$.

Notice also that $S \subset F \subseteq A_0$. On the other side, $A_0(A_0\{x\}) = A_0\{x\}$ and $A_0(A_0\{b\}) = A_0\{b\}$ for each $x \in M$, $b \in B$. For each j, k, l in $\{0, \dots, 2^n - 1\}$ the minimal subalgebra $A_{(j,k,l)}$ in B generated by $\{A_0, i_j, i_k, i_l\}$ is alternative (see [1,4,9]). Therefore, $i_j(i_ki_l) + i_k(i_ji_l) = 0$ for each $j \ne k$ with $j \ge 1$ and $k \ge 1$, l in $\{0, \dots, 2^n - 1\}$, since $(i_j + i_k)((i_j + i_k)i_l) = ((i_j + i_k)(i_j + i_k))i_l$, $i_ji_k + i_ki_j = 0$, $i_j(i_ji_l) = (i_ji_j)i_l$. Then $i_k\beta = U_k\beta$ with $2^n \times 2^n$ matrix U_k with entries in S for each k. From this and Conditions (1.1)–(1.4) in Definition 1.2, and $x = \beta[x]$ for each $x \in {}_B\hat{M}_B$, it follows that $span_{A_0}Q = (BD)^{<1>}$ and $span_{A_0}Q = span_{A_0}(GD)$, $span_{A_0}(GD) = span_{A_0}(Gspan_{A_0}(GD))$, since $D \subset {}_B\hat{M}_B$, $S \subset A_0$. It implies that $(BD)^{<2>} = (BD)^{<1>}$. By induction this gives $(BD)^{<m+1>} = (BD)^{<m>}$ for each $2 \le m \in \mathbf{N}$, hence $(BD)^{<\infty>} = (BD)^{<1>}$.

Certainly $(BD)^{<\infty>}$ is the left B-submodule in ${}_BN$, consequently, $(BD)^{<1>}$ is the left B-submodule in ${}_BN$. □

Corollary 1.6. *Let the conditions of Theorem 1.4 be satisfied. Then $\overline{(BD)^{<\infty>}} = (\bar{D}B)^{<\infty>}$.*

P r o p o s i t i o n 1.3. *Let the conditions of Theorem 1.1 be satisfied with $A_0 = F$, where F is a field, $char(F) \ne 2$. Let either $M_0 = F^m$ and $m \in \mathbf{N}$, or M_0 be a F-linear space such that $M_0 \ominus Fy$ be isomorphic with M_0 for each $y \in M_0$. Then for each $x \in M$ there exist an invertible F-linear operator $V : M \to M$ and $b \in B$ and $y \in M_0$ such that $Vx = by$.*

P r o o f. If $x = 0$ the assertion of this theorem is evident. For $x \ne 0$ in M there is the decomposition $x = x_0i_0 +, \dots, +x_{2^n-1}i_{2^n-1}$ with $x_0, \dots, x_{2^n-1}$ in M_0 such that there exists $k \in \{0, \dots, 2^n - 1\}$ with $x_k \ne 0$. So it is possible to choose such marked k. If $M_0 = F^m$, then it has a basis $e_1, \dots, e_m$ as the F-linear space. Therefore, for each $0 \ne x_j \in M_0$ there exists an invertible F-linear operator V_{x_j} on M_0 such that $V_{x_j}x_j = x_k$. If M_0 is the F-linear space such that $M_0 \ominus Fy$ is isomorphic with M_0 for each $y \in M_0$, then for each

$0 \neq x_j \in M_0$ there exists an invertible F-linear operator V_{x_j} on M_0 such that $V_{x_j}x_j = x_k$. We put $V = \bigoplus_{j=0}^{2^n-1} \hat{V}_{x_j}$, where $V_{x_l} = id_{M_0}$ if $x_l = 0$ or if $l = k$, where $\hat{V}_{x_j} : M_0 i_j \to M_0 i_j$, $\hat{V}_{x_j}(y i_j) = (V_{x_j}(y)) i_j = i_j(V_{x_j}(y))$ for each $y \in M_0$. In view of Theorem 1.1 $M_0 = \mathsf{C}_M(B)$, hence it is naturally $V(bI) = (bI)V$ for each b in F, where $I = id_M$. Therefore, V is the left and right F-linear operator on M such that V is invertible on M, since $\mathsf{C}_B(B) = F$ in the considered case $A_0 = F$ with $n \geq 2$. This implies that $Vx = by$ with $y = x_k$ and $b = \sum_{j \in \Lambda_x} i_j$, where $\Lambda_x = \{j \in \{0, \dots, 2^n - 1\} : x_j \neq 0\}$. □

Corollary 1.7. *Let B be the division alternative algebra, let M be a B-bimodule with involution satisfying the conditions of Theorem 1.1, $x = by$ with $y \in M_0$, $b \in B$. Then*

$$(B\{x\})^{<\infty>} = (\{x\}B)^{<\infty>} = (B\{x\})^{<\infty,\infty>}.$$

1.1. Conclusion

The results of this paper can be used for further studies of a structure of modules over nonassociative algebras, operator theory in modules over Dickson algebras, their applications to PDEs, mathematical physics, quantum field theory, their applications in other sciences, etc.

This can be used for analysis and solution of PDEs utilized in gas dynamics and high energy density physics, hydrodynamics, particularly, describing tidal deformations and the gravitational potential of the planet [17, 19–21].

It is worth to mention, that spectral theory of operators over Dickson algebras and particularly Cayley algebras was studied in [15–17]. Therefore, using the results obtained in this article, it will be important to investigate further operator theory in modules over generalized Dickson algebras, theory of factors for nonassociative analogs of C^*-algebras, analogs of direct integrals for them, applications in coding theory [22], etc.

2. Appendix

D e f i n i t i o n 2.1. Let X be an algebra over a ring F, let M be a X-bimodule and $B \subseteq X$. We put

$$Com_M(B) := \{x \in M : \forall b \in B, xb = bx\};$$

$$N_{M,l}(B) := \{x \in M : \forall b \in B, \forall c \in B, (xb)c = x(bc)\};$$

$$N_{M,m}(B) := \{x \in M : \forall b \in B, \forall c \in B, (bx)c = b(xc)\};$$

$$N_{M,r}(B) := \{x \in M : \forall b \in B, \forall c \in B, (bc)x = b(cx)\};$$

$$N_M(B) := N_{M,l}(B) \cap N_{M,m}(B) \cap N_{M,r}(B) \text{ and}$$

$$\mathsf{C}_M(B) := Com_M(B) \cap N_M(B).$$

Then $Com_M(B)$, $N_M(B)$, and $\mathsf{C}_M(B)$ are called a commutant, a nucleus and a centralizer correspondingly of the X-bimodule M relative to a subset B in X. Instead of $Com_M(X)$, $N_M(X)$, or $\mathsf{C}_M(X)$ it will be also written shortly Com_M, N_M, or C_M correspondingly.

A left (or right) X-module M is also denoted by ${}_XM$ (or M_X correspondingly), similarly for bimodules.

E x a m p l e 2.1. Particularly over the real field $F = A_0 = \mathbf{R}$ for $A_r(f_{(r)})$, $2 \leq r$, up to normalization of the doubling generator $\mathbf{l}_k$ on k-th step, a scalar $f_k \in \{-1, 1\}$ can be chosen for each $k = 1, 2, \dots$ (see Definition 1.1 and Remark 1.1). Frequently $\bar{a}$ is also denoted by a^* or $\tilde{a}$.

D e f i n i t i o n 2.2. Let N and M be two left B-modules (see Definition 1.2). A map $T : N \to M$ we call a left B-quasi-linear operator, if it is additive:

$$T(v + w) = T(v) + T(w)$$

and left $\mathsf{C}_B(B)$-homogeneous:

$$T(av) = aT(v)$$

for each $a \in \mathsf{C}_B(B)$, v and $w \in N$.

Evidently, each left B-quasi-linear operator is left $\mathsf{C}_B(B)$-linear. Similarly right B-quasi-linear operators for right B-modules are defined. If N and M are B-bimodules and a map $T : N \to M$ is left and right B-quasi-linear, then T will be called a B-quasi-linear operator.

If for left B-modules N and M the operator T is additive and

$$T(bv) = bT(v)$$

for each $b \in B$, v in N, then it will be called left B-linear. Analogously right B-linear operators for right B-modules are defined. If N and M are B-bimodules and a map $T : N \to M$ is left and right B-linear, then T will be called a B-linear operator.

The operator left or right B-quasi-linear (or left or right B-linear) $T : M \to M$ is called invertible if there exists a left or right B-quasi-linear (or left or right B-linear correspondingly) operator $V : M \to M$ such that $TV = I$ and $VT = I$, where $I = id_M$, where $id_M(x) = x$ for each $x \in M$. Then V is called an inverse operator of T and also denoted by T^{-1}.

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Information about the author

Sergey V. Ludkovsky, Doctor of Physical and Mathematical Sciences, Professor of Applied Mathematics Department, MIREA – Russian Technological University, Moscow, Russian Federation. E-mail: sludkowski@mail.ru
ORCID: https://orcid.org/0000-0002-4733-8151

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Информация об авторе

Людковский Сергей Викторович, доктор физико-математических наук, профессор кафедры прикладной математики, МИРЭА – Российский технологический университет, г. Москва, Российская Федерация. E-mail: sludkowski@mail.ru
ORCID: https://orcid.org/0000-0002-4733-8151

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