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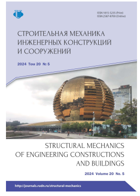

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Algorithm for Calculating Statically Indeterminate Trusses Using the Force Method

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Abstract. The study focuses on developing an algorithm for calculating statically indeterminate trusses using the force method. The main challenge in algorithmizing the force method lies in obtaining the solution to the homogeneous equilibrium equations, which is complicated by the ambiguity in selecting the primary system. The idea behind the presented algorithm is based on using the transposed compatibility matrix of the structure as the general solution to the homogeneous equilibrium equations. The governing system of equations eliminates the need to select redundant unknowns, as the column of unknowns is generated automatically. The method for obtaining compatibility equations in statically indeterminate truss cells is presented through a direct examination of changes in the area of truss loops. The compatibility matrix of the system is composed of rows of compatibility equations for independent statically indeterminate truss loops. Compatibility equations for the deformations of triangular and rectangular truss cells are derived, and a method for obtaining compatibility equations for externally statically indeterminate trusses is described. Using the proposed algorithm, the flexibility matrix of a truss with parallel chords is presented. The algorithm removes the ambiguity in selecting the primary system, and the structure of the flexibility matrix is determined by the numbering of the statically indeterminate loops of the system. There is no need to use the equilibrium equations when constructing the flexibility matrix of the structure.

Keywords: planar truss, general solution of the equilibrium equations, strain compatibility equations, continuity conditions of the area, forse method algorithm, flexibility matrix

Conflicts of interest. The authors declare that there is no conflict of interest.

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Алгоритм метода сил в расчетах статически неопределимых ферм

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Аннотация. Работая посвящена построению алгоритма расчёта статически неопределимых ферм методом сил. Основной трудностью в алгоритмизации метода сил является построение общего решения однородных уравнений равновесия, что объясняется неоднозначностью выбора основной системы. Идея излагаемого алгоритма основана на использовании транспонированной матрицы совместности деформации конструкции в качестве общего решения однородных уравнений равновесия узлов конструкции. Построенная система разрешающих уравнений позволяет отказаться от выбора лишних неизвестных, столбец неизвестных формируется автоматически. Изложен метод получения уравнений совместности деформаций ячеек статически неопределимых ферм с помощью рассмотрения изменения площади контуров ячейки. Матрица совместности деформаций системы составляется из строк уравнений совместности деформаций независимых статически неопределимых ячеек фермы. Получены уравнения совместности деформаций треугольной и прямоугольной ячеек ферм, изложен метод построения уравнений совместности деформаций для внешне статически неопределимых ферм. С использованием изложенного алгоритма приведена матрица податливости конструкции фермы с параллельными поясами с крестовой решёткой. Изложенный алгоритм снимает неоднозначность выбора основной системы, структура матрицы податливости конструкции однозначно определяется нумерацией статически неопределимых контуров системы. Для построения матрицы податливости конструкции нет необходимости использования уравнений равновесия узлов.

Ключевые слова: ферма, общее решение уравнений равновесия, уравнения совместности деформаций, условия неразрывности площади, метод сил, матрица податливости

Заявление о конфликте интересов. Авторы заявляют об отсутствии конфликта интересов.

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1. Introduction

The duality of the displacement method and the force method in structural mechanics is well known, and the application of the methods for “manual” analysis structures is approximately equally labor-intensive. There are certain classes of problems where one or another method may be convenient, for example, in terms of the number of unknowns, but the methods can be considered to be on the same footing.

However, the equality of the methods is lost when CAE packages are used to analyze structures. The absolute majority of commercial software packages are based on the displacement method. The advantage of the displacement method is the relative simplicity of its algorithmization, the matrix of governing equations is unambiguously determined by the numbering of the structure nodes. At the same time, the stiffness matrix has a band structure, is sparsely populated and, generally, is well-conditioned.

In contrast, the matrix of the governing equations of the force method can be formed in a non-unique way. From the point of view of classical structural mechanics, this is explained by the non-uniqueness of the choice of the primary system.

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In turn, the force method is important in optimization problems and exhibits efficiency in comparison with the displacement method [1–3], nonlinear analysis of structures [4–6], analysis of prestressed structures [7], and design of adaptive structures [8].

The known ways of algorithmization of the force method can be divided into three main groups: algebraic, topological and mixed.

Algebraic methods are generally reduced to particular operations on the matrix of nodal equilibrium equations. One of the first attempts to construct an algorithm for selecting redundant unknowns was the use of the Gauss-Jordan method [9; 10]. Subsequently, LU decomposition methods [11–13], singular value decomposition [14; 15] were proposed, as well as one of the methods for constructing the general solution presented in [16]. The mixed algebraic-topological methods are discussed in papers [17–20].

The main disadvantage of the algebraic methods is the necessity of preliminary application of complex operations on the matrix of equilibrium equations of the structure to form the matrix of governing equations of the force method. This disadvantage prevents from constructing an efficient algorithm in terms of the number of computational operations.

Topological methods are based on the use of the geometric properties of the structure, such as periodicity and cyclicity. The methods proposed in [21–23] can be referred to topological methods. One of the varieties of such methods is discussed in studies [24–26] devoted to the use of the fundamental basis of cycles of a graph, which is equivalent for the considered structure. Algorithms that exploit the cyclic nature of the structure have been proposed [27; 28]. The issue with the methods based on periodicity or cyclicity of the structure is that they cannot be applied to problems with arbitrary geometry. The use of graph operations has the same disadvantage as the algebraic methods.

The widely used integrated force method, first proposed in [29], can be highlighted. Currently, the integrated force method has been generalized to plane and spatial problems of elasticity theory and nonlinear problems [30–32]. The key idea of the method is to solve the joint system of equilibrium equations of the structure and strain compatibility equations. However, the structure of the obtained matrix and the number of unknowns do not indicate efficiency of the method in comparison with the displacement method.

Thus, no algorithm for the force method comparable in complexity to the displacement method has been constructed so far.

This paper presents a method for the analysis of statically indeterminate trusses. The key idea is to use the transposed strain compatibility matrix as the matrix of general solution of the homogeneous equilibrium equations.

2. Method

2.1. Problem Statement of Force Method Algorithmization

The equations of structural mechanics of trusses can be written in the form of the following system of equations:

$$A^T N = P, \tag{1a}$$

$$AU = \varepsilon = \varepsilon^0 + \varepsilon^e, \tag{1b}$$

$$\varepsilon^e = \Lambda N, \tag{1c}$$

where A^T is the specified nodal equilibrium matrix; $[...]^T$ is the matrix transpose operation; N is the column of axial forces in the truss members; P is the column of specified nodal loads; U is the column of nodal displacements; ε is the column of axial strains of the members; ε^0 is the column of specified initial strains of the members; ε^e is the column of elastic strains of the members; $\Lambda = diag\left(l_i / EA_i\right)$ is the flexibility coefficient matrix of the members of the system; l_i is the length of the *i*-th member; EA_i is the axial stiffness of the *i*-th member.

Equation (1a) represents the equilibrium equations of the system nodes, (1b) represents the geometric equations relating displacements and deformations, and (1c) represents the constitutive equations relating forces and deformations.

It is known that the general solution of a non-homogeneous system of equations is the sum of some particular solution of this system and the general solution of the corresponding homogeneous system of equations.

In statically indeterminate systems, the rank of matrix A^T is equal to the number of its rows and is obviously less than the number of unknowns, and therefore, the system of the homogeneous equilibrium equations has a nontrivial solution. The construction of the general solution is the main difficulty in the algorithmization of the force method.

Suppose that the fundamental system of solutions of the homogeneous system is constructed. The columns of the fundamental system are taken as the rows of some matrix B. By definition of the fundamental system:

$$A^T B^T = 0. \tag{2}$$

Therefore, for an arbitrary column F the following is valid:

$$A^T B^T F = 0. \tag{3}$$

Thus, $B^T F$ is the general solution of the system of the homogenous equilibrium equations. Considering an arbitrary particular solution N_p and (1a):

$$N = B^T F + N_P. \tag{4}$$

By transposing (2), one obtains:

$$BA = 0. \tag{5}$$

Multiplying (1b) by B yields:

$$BAU = B\varepsilon = B\left(\varepsilon^0 + \varepsilon^e\right), \tag{6}$$

and taking into account (5), the following is valid for any column U:

$$B\left(\varepsilon^0 + \varepsilon^e\right) = 0. \tag{7}$$

By substituting (3) into (1c), one obtains:

$$\varepsilon^e = \Lambda B^T F + \Lambda N_p. \tag{8}$$

Substituting (8) into (7) yields the governing system of equations of the force method:

$$B\Lambda B^T F + B\varepsilon^0 + B\Lambda N_p = 0. \tag{9}$$

Similar to the method of displacements, $B\Lambda B^T$ is the flexibility matrix of the structure. The solution of the problem is now reduced to the solution of system (9), the forces in the structural elements are recalculated according to (4).

Expression (7) has the physical meaning of the strain compatibility equations. From the algebraic point of view, the transpose of compatibility matrix B produces the desired general solution of the homogeneous equilibrium equations. This is the essence of the proposed method, as it will be shown later, the strain compatibility equations can be constructed without using the nodal equilibrium matrix.

The physical meaning of the unknown column F in system (9) remains unknown. There is no need to choose the primary system and "extra" unknowns, the vector of unknowns is formed "automatically".

2.2. Construction of Strain Compatibility Equations

The idea behind the proposed method of constructing the strain compatibility equations is the relationship between the strain of particular members constituting a loop and the change in area of this loop. For illustration, a truss cell, which is statically indeterminate to the first degree, is shown in Figure 1, *a*. Here, the members are numbered, arrows indicate their orientation, and letters i,k,m,s denote the nodes of the structure. It should be noted that the numbering and orientation of the members do not affect the final result.

The cell under consideration consists of three independent loops 1, 2, 3 denoted in Figure 1, *b*. These three loops together constitute the fourth one, shown in Figure 1, *c*.

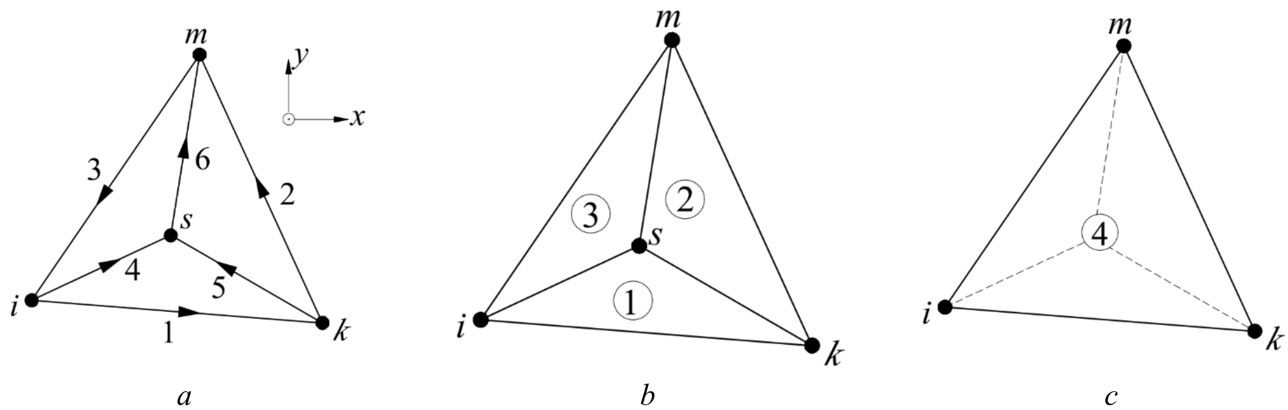

Figure. 1. Truss diagram:
a — numbering of members and nodes; *b* — loops No. 1, 2, 3; *c* — loop No. 4
S o u r c e : made by V.V. Lalin, T.R. Ibragimov

The following relationship is valid for the areas of the considered loops:

$$S_4 = S_1 + S_2 + S_3, \tag{10}$$

where S_j is the area of the *j*-th loop.

After deformation of the structure due to external loads, the areas of the loops will change, but for the new values of the areas the same identity will be true due to the continuity of the structure:

$$S_4' = S_1' + S_2' + S_3'. \tag{11}$$

By denoting the change in area of the *j*-th loop as $\Delta S_j = S_j' - S_j$, the following relationship is obtained:

$$\Delta S_4 = \Delta S_1 + \Delta S_2 + \Delta S_3. \tag{12}$$

Expression (12) has the meaning of the continuity condition of the loop area. If expressed through the member strains, equation (12) will be the desired equation of strain compatibility of the considered truss cell.

3. Results and Discussion

3.1. Strain Compatibility Equation of 6-member and 4-node Cell

Before obtaining the strain compatibility equations of the truss cell in Figure 1, an arbitrary member in x, y plane and nodes i and k is considered. The member is oriented by unit vector $t^T = \left[t_x, t_y\right]$. Unit vector $n^T = \left[n_x, n_y\right]$, which is normal to vector t, is introduced such that vectors t, n, z constitute a right-hand vector system, similar to coordinate system x, y, z.

The nodal displacements of the member are written as $U_i^T = \left[U_{ix}, U_{iy}\right]$, $U_k^T = \left[U_{kx}, U_{ky}\right]$. Axial strains of the member can now be expressed as:

$$\varepsilon = t^T\left(U_k - U_i\right). \tag{13}$$

The following notation is introduced:

$$\omega = n^T\left(U_k - U_i\right). \tag{14}$$

Thus, ω represents the relative displacement along the normal to the axis of the member, that is, the relative displacement of the nodes corresponding to the rotation of the member as a rigid body.

The following expression follows from equations (13), (14), which relates the member strains and the displacements of its nodes:

$$U_k - U_i = \varepsilon t + \omega n. \tag{15}$$

A convenient tool for evaluating the change in area is the outer product operation [33]. The outer product of two vectors $a = \left[a_x, a_y\right]^T$, $b = \left[b_x, b_y\right]^T$ lying in the x, y plane can be written as:

$$a \wedge b = \det\left[a, b\right] = \begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix}, \tag{16}$$

where $\det[\ldots]$ is the matrix determinant.

The main properties of the outer product [33]:

$$\begin{aligned} &a \wedge b = -b \wedge a, \\ &a \wedge (\lambda b) = \lambda(a \wedge b) = (\lambda a) \wedge b, \quad \lambda \in \mathbb{R}, \\ &a \wedge (b + c) = a \wedge b + a \wedge c, \\ &a \wedge b = 0 \leftrightarrow a \parallel b, \quad a, b \neq 0. \end{aligned} \tag{17}$$

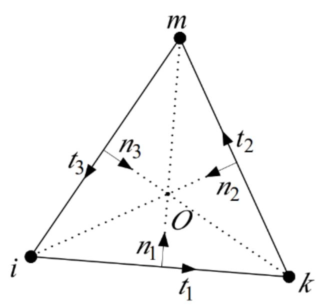

Figure. 2. Loop No. 4
Source: made by V.V. Lalin, T.R. Ibragimov

The outer product is the oriented area of the parallelogram constructed with the multiplied vectors, that is, it is equal to the area of the parallelogram with a positive or negative sign depending on whether the axis triples x, y, z and a, b, z coincide in orientation or not.

Now loop 4 is examined to determine its change in area. Unit vectors for each member (Figure 2) are introduced.

Let r_i, r_k, r_m be the position vectors of the nodes of loop i, k, m having an arbitrary origin. By using outer product, the change in area may be written as:

$$2\Delta S_4 = S_4 - S_4^{'} = \left(r_k - r_i\right) \wedge \left(r_m - r_i\right) - \left(r_k + U_k - r_i - U_i\right) \wedge \left(r_m + U_m - r_i - U_i\right). \quad (18)$$

By expanding the brackets and grouping the terms, the following is obtained:

$$2\Delta S_4 = U_i \wedge U_k + U_k \wedge U_m + U_m \wedge U_i + \left(U_m - U_k\right) \wedge r_i + \left(U_k - U_i\right) \wedge r_m + \left(U_i - U_m\right) \wedge r_k. \quad (19)$$

It should be noted that the second group of terms consists of the products of the difference between the displacements of the nodes and the position vector of the opposite node. The strains of each of the three members of the loops are expressed according to (15):

$$\begin{aligned} U_k - U_i &= \varepsilon_1 t_1 + \omega_1 n_1; \\ U_m - U_k &= \varepsilon_2 t_2 + \omega_2 n_2; \\ U_i - U_m &= \varepsilon_3 t_3 + \omega_3 n_3. \end{aligned} \quad (20)$$

Thus, expression (19) will include the products of the member strains (and their rotation as a rigid body) and the position vectors of the opposite nodes. According to the properties of the outer product, the rotation component can be eliminated by requiring the position vector and vector n normal to the member to be parallel. For any triangular loop this is possible if the orthocenter of the triangle (point O in Figure 2) is taken as the origin of the position vector. By rewriting (19), leaving the non-zero terms, one obtains:

$$2\Delta S_4 = U_i \wedge U_k + U_k \wedge U_m + U_m \wedge U_i + \varepsilon_1 t_1 \wedge r_m^{ikm} + \varepsilon_2 t_2 \wedge r_i^{ikm} + \varepsilon_3 t_3 \wedge r_k^{ikm}, \quad (21)$$

where the upper index ikm denotes that the origin of the position vector is the orthocenter of triangle ikm.

Similarly, the changes in area of loops 1, 2, 3 are determined:

$$2\Delta \mathrm{S}_1 = U_s \wedge U_k + U_k \wedge U_i + U_i \wedge U_s + \varepsilon_1 t_1 \wedge r_s^{ski} - \varepsilon_4 t_4 \wedge r_k^{ski} + \varepsilon_5 t_5 \wedge r_i^{ski}; \quad (22a)$$

$$2\Delta \mathrm{S}_2 = U_s \wedge U_m + U_m \wedge U_k + U_k \wedge U_s + \varepsilon_2 t_2 \wedge r_s^{smk} - \varepsilon_5 t_5 \wedge r_m^{smk} - \varepsilon_6 t_6 \wedge r_k^{smk}; \quad (22b)$$

$$2\Delta S_3 = U_s \wedge U_m + U_m \wedge U_i + U_i \wedge U_s + \varepsilon_3 t_3 \wedge r_s^{smi} + \varepsilon_4 t_4 \wedge r_m^{smi} + \varepsilon_6 t_6 \wedge r_i^{smi}. \quad (22c)$$

Substituting now (21)–(22) into (12), it can be seen that the quadratic displacement terms are identically eliminated. The obtained expression will be the strain compatibility equation for a statically indeterminate to the first degree truss cell:

$$\begin{aligned} &\varepsilon_1 t_1 \wedge r_m^{ikm} + \varepsilon_2 t_2 \wedge r_i^{ikm} + \varepsilon_3 t_3 \wedge r_k^{ikm} = \\ &= \varepsilon_1 t_1 \wedge r_s^{ski} - \varepsilon_4 t_4 \wedge r_k^{ski} + \varepsilon_5 t_5 \wedge r_i^{ski} + \\ &+ \varepsilon_2 t_2 \wedge r_s^{smk} - \varepsilon_5 t_5 \wedge r_m^{smk} - \varepsilon_6 t_6 \wedge r_k^{smk} + \\ &+ \varepsilon_3 t_3 \wedge r_s^{smi} + \varepsilon_4 t_4 \wedge r_m^{smi} + \varepsilon_6 t_6 \wedge r_i^{smi}. \end{aligned} \quad (23)$$

Expression (23) is valid for any nondegenerate cell consisting of 6 members connected in a similar way to the considered case. Thus, for the cell shown in Figure 3, the outer product for the terms of the second loop will be obtained with a negative sign and expression (12) will be reduced to the following form:

$$\Delta S_{ikm} + \Delta S_{ksm} = \Delta S_{iks} + \Delta S_{ism}. \quad (24)$$

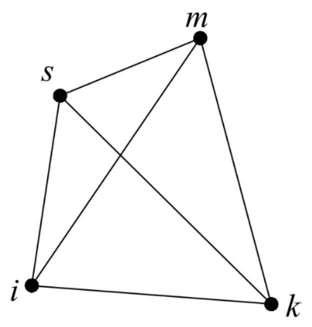

Figure 3. Topologically similar statically indeterminate cell
S o u r c e: made by V.V. Lalin, T.R. Ibragimov

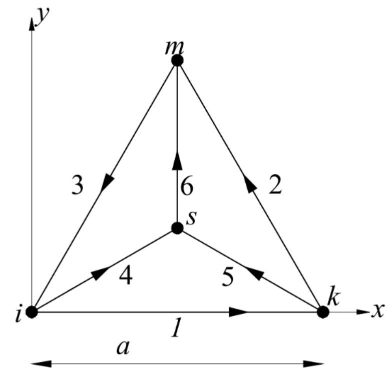

Figure 4. Equilateral triangle truss panel
S o u r c e: made by V.V. Lalin, T.R. Ibragimov

Strain compatibility matrix B can now be constructed from the rows of the strain compatibility equations for each statically indeterminate elementary cell. Thus, structure flexibility matrix $L = B\Lambda B^T$ is uniquely determined by the numbering of statically indeterminate loops of the system. At the same time, it is not necessary to use matrix A^T of nodal equilibrium equations to construct the structure flexibility matrix.

The algorithm of analysis using the force method comes down to the construction of strain compatibility equations for independent statically indeterminate cells in order to form the strain compatibility matrix of the system, construct of the structure flexibility matrix and solve the governing system.

Figure 4 demonstrates a structure in the form of an equilateral triangle with base a and node s in the center of mass of triangle ikm. Construction of the strain compatibility equations for this system is presented below.

Unit vectors t_i for the members are expressed as:

$$t_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ t_2 = \frac{1}{2}\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix},\ t_3 = -\frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix},$$

$$t_4 = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix},\ t_5 = \frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix},\ t_6 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The position vectors originating from the orthocenter and pointing to the nodes of loop 4:

$$r_i^{ikm} = -a\frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3}/3 \end{bmatrix},\ r_k^{ikm} = a\frac{1}{2}\begin{bmatrix} -1 \\ \sqrt{3}/3 \end{bmatrix},\ r_m^{ikm} = a\begin{bmatrix} 0 \\ \sqrt{3}/3 \end{bmatrix}.$$

As a result of calculating the outer products, the following is obtained:

$$\varepsilon_1 t_1 \wedge r_m = \varepsilon_1 \frac{a}{\sqrt{3}}, \varepsilon_2 t_2 \wedge r_i = \varepsilon_2 \frac{a}{\sqrt{3}}, \varepsilon_3 t_3 \wedge r_k = \varepsilon_3 \frac{a}{\sqrt{3}}.$$

Thus, the left-hand side of expression (12):

$$\frac{a}{\sqrt{3}}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3).$$

Similar procedure is applied to loops 1, 2, 3 to obtain:

loop iks: $\qquad a\left(\varepsilon_1 + \varepsilon_5 - \frac{1}{\sqrt{3}}\varepsilon_1\right),$

loop kms: $$a\left(\varepsilon_5+\varepsilon_6-\frac{1}{\sqrt{3}}\varepsilon_2\right),$$

loop ism: $$a\left(\varepsilon_4+\varepsilon_6-\frac{1}{\sqrt{3}}\varepsilon_3\right).$$

By substituting the obtained expressions into (23), expanding the brackets and grouping the terms, the strain compatibility equation is derived:

$$a\left[\varepsilon_1+\varepsilon_2+\varepsilon_3-\sqrt{3}\left(\varepsilon_4+\varepsilon_5+\varepsilon_6\right)\right]=0. \tag{25}$$

3.2. Cross Brace Truss

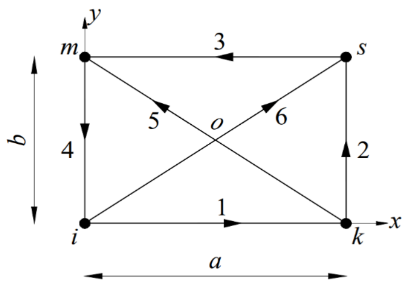

Figure 5. Rectangular truss panel
S o u r c e: made by V.V. Lalin, T.R. Ibragimov

As mentioned earlier, expression (23) is suitable for any cell with 6 members, however, there is an important degenerate case for which equation (23) is not acceptable.

Consider the design shown in Figure 5. The requirement of taking the orthocenter of the triangle as the origin of the position vector makes the terms accounting for the strains of members 5 and 6 equal to zero, since the orthocenter of a right triangle is at the apex of a right angle.

The strain compatibility equation can be obtained from the following equality for the areas:

$$\Delta S_{iks}+\Delta S_{ism}=\Delta S_{iko}+\Delta S_{kso}+\Delta S_{mos}+\Delta S_{iom}. \tag{26}$$

In this case, the following equalities must be used:

$$U_m-U_o=U_o-U_k=\frac{1}{2}(U_m-U_k);$$

$$U_i-U_o=U_o-U_s=\frac{1}{2}(U_i-U_s). \tag{27}$$

For the orientation of the members shown in Figure 5, the following unit vectors are used:

$$t_1=\begin{bmatrix}1\\0\end{bmatrix}, t_2=\begin{bmatrix}0\\1\end{bmatrix}, t_3=\begin{bmatrix}-1\\0\end{bmatrix};$$

$$t_4=\begin{bmatrix}0\\-1\end{bmatrix}, t_5=\frac{1}{c}\begin{bmatrix}-a\\b\end{bmatrix}, t_6=\frac{1}{c}\begin{bmatrix}a\\b\end{bmatrix},$$

where $c=\sqrt{a^2+b^2}$.

For the loop constructed with vertices i,k,s the orthocenter is point k, therefore

$$r_i=-\begin{bmatrix}a\\0\end{bmatrix}, r_s=\begin{bmatrix}0\\b\end{bmatrix},$$

and to the nearest quadratic terms:

$$2\Delta S_{iks} = \varepsilon_2 t_2 \wedge r_i + \varepsilon_1 t_1 \wedge r_s = \varepsilon_2 a + \varepsilon_1 b.$$

Similarly, for loop ism:

$$2\Delta S_{ism} = \varepsilon_4 a + \varepsilon_3 b.$$

For the loop with vertices i, k, o:

$$r_o = \frac{b^2 - a^2}{2b}\begin{bmatrix}0\\1\end{bmatrix},\ r_i = -\frac{a}{2}\begin{bmatrix}1\\a/b\end{bmatrix},\ r_k = \frac{a}{2}\begin{bmatrix}1\\-a/b\end{bmatrix};$$

$$2\Delta S_{iko} = \varepsilon_1 t_1 \wedge r_o + \frac{1}{2}\varepsilon_5 t_5 \wedge r_i + \frac{1}{2}\varepsilon_6 t_6 \wedge r_k = \frac{a \cdot c}{2b}\left(\varepsilon_5 + \varepsilon_6\right) - \varepsilon_1 \frac{b^2 - a^2}{2b}.$$

Similarly, the change in area of the loop with vertices m, o, s,

$$2\Delta S_{mos} = \frac{a \cdot c}{2b}\left(\varepsilon_5 + \varepsilon_6\right) - \varepsilon_3 \frac{b^2 - a^2}{2b}.$$

For the loop with vertices i, o, m:

$$r_o = \frac{b^2 - a^2}{2a}\begin{bmatrix}1\\0\end{bmatrix},\ r_i = -\frac{b}{2}\begin{bmatrix}b/a\\1\end{bmatrix},\ r_m = \frac{b}{2}\begin{bmatrix}-b/a\\1\end{bmatrix};$$

$$2\Delta S_{iom} = \varepsilon_4 t_4 \wedge r_o - \frac{1}{2}\varepsilon_6 t_6 \wedge r_m + \frac{1}{2}\varepsilon_5 t_5 \wedge r_i = \frac{b \cdot c}{2a}\left(\varepsilon_5 + \varepsilon_6\right) - \varepsilon_4 \frac{a^2 - b^2}{2a}.$$

Similarly, for loop kos:

$$2\Delta S_{kos} = \frac{b \cdot c}{2a}\left(\varepsilon_5 + \varepsilon_6\right) - \varepsilon_2 \frac{a^2 - b^2}{2a}.$$

After substitution into condition (26), the following strain compatibility equation is obtained after simplifications:

$$a\left(\varepsilon_1 + \varepsilon_3\right) + b\left(\varepsilon_2 + \varepsilon_4\right) - \sqrt{a^2 + b^2}\left(\varepsilon_5 + \varepsilon_6\right) = 0. \tag{28}$$

In the particular case of a square cell $(a = b)$, the equation becomes:

$$a\left[\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \sqrt{2}\left(\varepsilon_5 + \varepsilon_6\right)\right] = 0. \tag{29}$$

A similar expression is given in [34], where it was obtained by analyzing the matrix of nodal equilibrium equations of the structure.

Using the obtained expression (29), the formation of the structure flexibility matrix of the example truss presented in Figure 6 is discussed below. The truss consists of n square cells, the axial stiffness of each member is EA. The members are numbered according to the scheme shown in Figure 6. The total number of members in the truss is $1 + 5n$, the total number of nodes is $2(n+1)$.

The diagonal matrix of the member flexibility coefficients:

$$\Lambda = \frac{a}{EA}\operatorname{diag}\left[1,1,1,1,\sqrt{2},\sqrt{2},1,\ldots,1,\sqrt{2},\sqrt{2}\right].$$

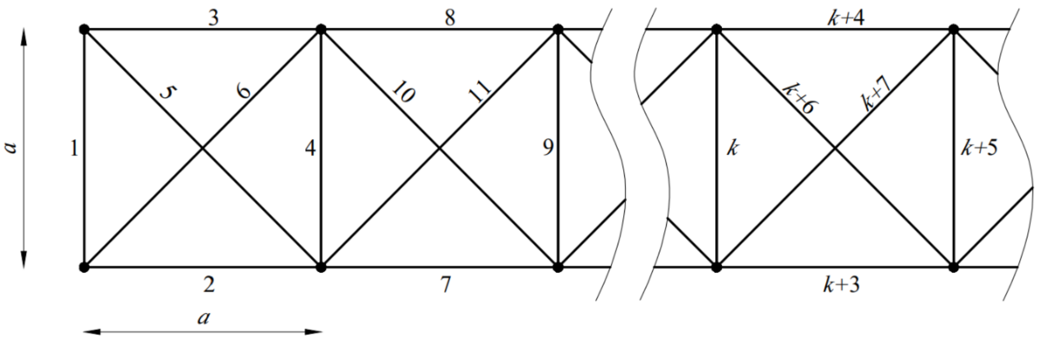

Figure 6. Rectangular truss
S o u r c e: made by V.V. Lalin, T.R. Ibragimov

The strain compatibility matrix of the system, according to (29), will have the following form (only the first three rows are shown):

$$B = a\begin{bmatrix} 1 & 1 & 1 & 1 & -\sqrt{2} & -\sqrt{2} & 0 & \cdots & & & & & & & & & \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & -\sqrt{2} & -\sqrt{2} & 0 & \cdots & & & & \\ & & & & & & \cdots & 0 & 1 & 0 & 0 & 1 & 1 & 1 & -\sqrt{2} & -\sqrt{2} & 0 & \cdots \end{bmatrix}.$$

By multiplying out $B\Lambda B^T$, the following tridiagonal flexibility matrix of the system is obtained:

$$L = \frac{a^3}{EA}\begin{bmatrix} 4\left(1+\sqrt{2}\right) & 1 & & \\ 1 & 4\left(1+\sqrt{2}\right) & 1 & \\ & & \ddots & \\ & & 1 & 4\left(1+\sqrt{2}\right) \end{bmatrix}. \quad (30)$$

Thus, the obtained flexibility matrix has the size of $n \times n$. The displacement method stiffness matrix, in turn, will have the dimension of $4(n+1)$.

3.3. Externally Statically Indeterminate Trusses

In this section, the problem of composing the strain compatibility equations for externally statically indeterminate trusses is considered. These are trusses, the support reactions of which cannot be determined from the equilibrium equations.

Figure 7, *a* shows a statically indeterminate to the first degree cell with two independent loops 1 and 2, and the third loop denoted in Figure 7, *b*.

An obvious equality is true for the areas of the loops:

$$\Delta S_3 = \Delta S_1 + \Delta S_2. \quad (31)$$

In the case of no additional support, the cell would be statically determinate and the quadratic displacement terms in equation (31) would not reduce.

The presence of supports leads to some constraints on the displacements of the nodes, as a result of which the quadratic terms are reduced and the equality can be expressed through the member strains.

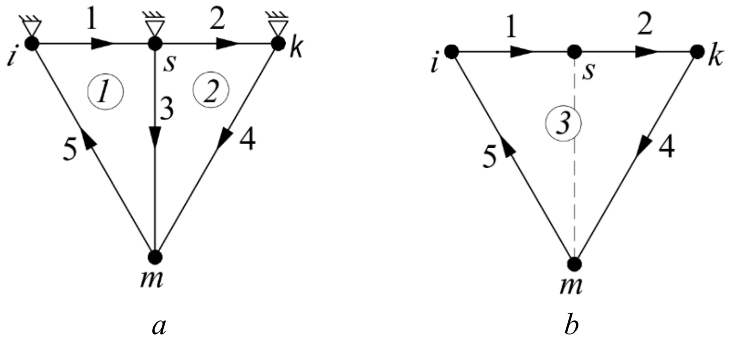

Figure 7. Externally statically indeterminate truss:
a — loops No. 1, 2; *b* — loop No. 3
S o u r c e: made by V.V. Lalin, T.R. Ibragimov

The changes in area of the loops taking into account that $U_i - U_k = U_i - U_s + U_s - U_k$:

$$2\Delta S_3 = U_i \wedge U_m + U_m \wedge U_k + U_k \wedge U_i + \\ +(U_k - U_m) \wedge r_i^{mik} + (U_m - U_i) \wedge r_k^{mik} + (U_s - U_k) \wedge r_m^{mik}; \quad (32a)$$

$$2\Delta S_1 = U_i \wedge U_m + U_m \wedge U_s + U_s \wedge U_i + \\ +(U_s - U_m) \wedge r_i^{ism} + (U_m - U_i) \wedge r_s^{ism} + (U_i - U_s) \wedge r_m^{ism}; \quad (32b)$$

$$2\Delta S_2 = U_m \wedge U_k + U_k \wedge U_s + U_s \wedge U_m + \\ +(U_s - U_k) \wedge r_m^{skm} + (U_m - U_s) \wedge r_k^{skm} + (U_m - U_k) \wedge r_s^{skm}. \quad (32c)$$

For the given cell, the displacements of nodes s, k are parallel, hence by the properties of the outer product:

$$U_s \wedge U_k = 0 . \quad (33)$$

In turn, the displacement of node i is zero and the remaining non-zero terms are:

$$2\Delta S_3 = U_i \wedge U_m + (U_k - U_m) \wedge r_i^{mik} + (U_m - U_i) \wedge r_k^{mik} + (U_s - U_k) \wedge r_m^{mik} ; \quad (34a)$$

$$2\Delta S_1 = U_m \wedge U_s + (U_s - U_m) \wedge r_i^{ism} + (U_m - U_i) \wedge r_s^{ism} + (U_i - U_s) \wedge r_m^{ism} ; \quad (34b)$$

$$2\Delta S_2 = U_m \wedge U_k + U_s \wedge U_m + (U_s - U_k) \wedge r_m^{skm} + (U_m - U_s) \wedge r_k^{skm} + (U_m - U_k) \wedge r_s^{skm} . \quad (34c)$$

As seen from expressions (34), the quadratic terms are identically eliminated when substituted into expression (31). The remaining ones, written in terms of member strains in accordance with (15), represent the strain compatibility equation for the considered externally statically indeterminate truss.

By taking, for example, the lengths of members 1, 2, 3 equal to a, and correspondingly the lengths of members 4, 5 equal to $\sqrt{2}a$, it is possible to obtain the following strain compatibility equation using (31):

$$a\left[\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 - \sqrt{2}(\varepsilon_4 + \varepsilon_5)\right] = 0 . \quad (35)$$

4. Conclusion

1. The main problem in the algorithmization of the force method is finding the general solution to the homogeneous equilibrium equations of the structure $A^{\mathrm{T}}N = 0$. The method of obtaining the strain compatibility equations completes the construction of the algorithm for solving the problems of statically indeterminate trusses using the force method.

2. The proposed formulation of the force method allows to not have to select the "primary system" and the unknowns of the force method. The proposed method automatically "selects" the vector of unknowns F. The numbering of statically indeterminate loops unambiguously determines the structure of the flexibility matrix of the system.

3. The advantage of the proposed method is that the equilibrium equations of the structure are not required. There is no need to store in the computer memory and use the matrix of nodal equilibrium equations of the structure A^{T}.

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