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# Linear perturbations of the Bloch type of space-periodic magnetohydrodynamic steady states. I. Mathematical preliminaries

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**Abstract:** We consider Bloch eigenmodes in three linear stability problems: the kinematic dynamo problem, the hydrodynamic and MHD stability problem for steady space-periodic flows and MHD states. A Bloch mode is a product of a field of the same periodicity, as the state subjected to perturbation, and a planar harmonic wave, $\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}$. The complex exponential cancels out from the equations of the respective eigenvalue problem, and the wave vector $\mathbf{q}$ remains in the equations as a numeric parameter. The resultant problem has a significant advantage from the numerical viewpoint: while the Bloch mode involves two independent spatial scales, its growth rate can be computed in the periodicity box of the perturbed state. The three-dimensional space, where $\mathbf{q}$ resides, splits into a number of regions, inside which the growth rate is a smooth function of $\mathbf{q}$. In preparation for a numerical study of the dominant (i.e., the largest over $\mathbf{q}$) growth rates, we have derived expressions for the gradient of the growth rate in $\mathbf{q}$ and proven that, for parity-invariant flows and MHD steady states or when the respective eigenvalue of the stability operator is real, half-integer $\mathbf{q}$ (whose all components are integer or half-integer) are stationary points of the growth rate. In prior works it was established by asymptotic methods that high spatial scale separation (small $\mathbf{q}$) gives rise to the phenomena of the $\alpha$-effect or, for parity-invariant steady states, of the eddy diffusivity. We review these findings tailoring them to the prospective numerical applications.

**Keywords:** Kinematic dynamo problem, hydrodynamic linear stability problem, magnetohydrodynamic linear stability problem, Bloch mode, magnetic $\alpha$-effect, AKA-effect, combined magnetohydrodynamic $\alpha$-effect, magnetic eddy diffusivity, eddy viscosity, scale separation.

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## 1. Introduction

A central notion of the hydromagnetic dynamo theory is the magnetic $\alpha$-effect. It is built on the seminal idea of E. Parker, who suggested [*Parker*, 1955] that the interaction of small-scale fluctuations of the flow ("cyclonic events") and of magnetic field may give rise to an electromotive force that has a component parallel to the mean magnetic field, and it can amplify the mean field. A systematic analysis of this idea under various simplifying assumptions was carried out by German scientists [*Krause and Rädler*, 1980; *Steenbeck et al.*, 1971], who developed the theory of the mean-field magnetohydrodynamics and coined the term $\alpha$-effect (see [*Rädler*, 2007] for an account of the history of the subject).

In principle, any averaging procedure satisfying the Reynolds rules (see, e.g., [*Krause and Rädler*, 1980]) can be employed to define mean fields. Historically, the magnetic $\alpha$-effect in a weakly non-axisymmetric flow was the first example of the $\alpha$-effect derived rigorously by asymptotic methods from the first principles [*Braginskii*, 1964a,b]. An expression for the $\alpha$-effect in an almost axisymmetric fluid flow was obtained by S. I. Braginskii in the cylindrical coordinates by averaging in the azimuthal variable $\phi$. This model was used to explain how the geomagnetic dynamo operates [*Braginskii*, 1964c,d, 1975], and later, in a more general form of the $\alpha\omega$-dynamo, for explanation of the origin of the solar magnetic field (see [*Cameron et al.*, 2016; *Charbonneau*, 2005, 2014; *Miesch*, 2012]). A very careful tuning of the localization and amplitude of the $\alpha$-effect turned out to be necessary [*Ossendrijver*, 2000; *Popova*, 2016] to achieve this goal.

The mean magnetic field obtained by averaging based on separation of spatial scales (together with temporal ones, for time-periodic or quasiperiodic flows) also proved to be amenable to mathematical treatment from the first principles without additional assumptions. We focus attention on the $\alpha$-effect and eddy ("turbulent") diffusivity relying on this physical mechanism. A magnetic mode is supposed to depend on the so-called "fast", $\mathbf{x}$, (responsible for small scales) and "slow", $\mathbf{X} = \varepsilon\mathbf{x}$, (responsible for large scales) variables. We assume for simplicity that the small-scale (i.e., independent of slow variables) generating flow is steady, and the flow and magnetic modes are $2\pi$-periodic in each Cartesian axis $x_i$; the periodicity box in $\mathbf{x}$ is denoted by $\mathbb{T}^3 = [-\pi, \pi]^3$. The mean field is then understood as an average over the fast variables that depends on the slow variables,

$$\langle \mathbf{b}(\mathbf{x}, \mathbf{X}) \rangle = (2\pi)^{-3} \int_{\mathbb{T}^3} \mathbf{b}(\mathbf{x}, \mathbf{X})\, \mathrm{d}\mathbf{x}$$

(averaging over fast time is also necessary if the generating flow and the modes are periodic in the fast time). The scale ratio, $\varepsilon$, is regarded as a small parameter and used for construction of an asymptotic expansion of the mode and the associated eigenvalue of the magnetic induction operator in power series in this parameter. By this technique (known as homogenization of elliptic operators), it is possible to derive equations for the evolution of the mean field, expressions for the $\alpha$-effect tensor in terms of small-scale neutral magnetic modes [*Roberts*, 1970; *Vishik*, 1987] (see also [*Andrievsky et al.*, 2015; *Rasskazov et al.*, 2018]) and, when the $\alpha$-effect vanishes (e.g., when the generating flow $\mathbf{V}$ is parity-invariant, i.e., satisfies $\mathbf{V}(\mathbf{x}) = -\mathbf{V}(-\mathbf{x})$), for the magnetic eddy diffusivity tensor [*Lanotte et al.*, 1999; *Roberts*, 1972] (see also [*Zheligovsky*, 2011]).

This approach gave an opportunity to prove without recourse to numerics existence of large-scale dynamos employing the magnetic $\alpha$-effect. The $\alpha$-effect operator controlling the leading terms in the expansion of the eigenvalues has the spectrum symmetric about the imaginary axis [*Vishik*, 1987], and thus generation is guaranteed for sufficiently high scale separations unless the entire spectrum lies on this axis, which is not a generic case. Among other applications of the homogenization techniques, let us mention the dynamo for flows with an internal scale, where the limit operator is the sum of the $\alpha$-effect and molecular diffusivity operators [*Vishik*, 1986; *Zheligovsky*, 1991], and the weakly nonlinear convective dynamo [*Chertovskih and Zheligovsky*, 2015].

It is important that large-scale magnetic modes residing in the entire space that arise in the kinematic dynamo problem, turn out to have a special structure: they are small-scale (i.e., having the same periodicity as the generating flow) fields, amplitude-modulated by large-period harmonics, or, equivalently, they are products of the small-scale fields and planar harmonic waves, $\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}$, responsible for their large-scale shape (here $\mathbf{q}$ is a constant wave vector; the scale separation is high when $|\mathbf{q}| = \varepsilon > 0$ is small). This can be expected, because for small-scale flows the domain of the magnetic induction operator decomposes into a direct sum of invariant eigenspaces comprised of vector fields of this structure. Functions of this type emerged in the solid-state physics: by the Bloch theorem [*Bloch*, 1929], solutions to the Schrödinger equation with a space-periodic potential, that describe the state of an electron in a periodic crystal, have such a structure. The ansatz was widely used in the dynamo theory, e.g., it was systematically investigated by G. O. Roberts [*Roberts*, 1970, 1972], and used to introduce the axial non-symmetry of the magnetic modes in the study of the dynamo properties of the Couette–Poiseuille flow between coaxial cylinders [*Ruzmaikin et al.*, 1989; *Soloviev*, 1985a,b,c, 1987]. (In the latter case the flow depends on the radial coordinate $r$ only, and the exponential dependence $\mathbf{B}' = \mathbf{b}(r)\exp(\mathrm{i}(n\phi + \alpha z))$ on the coordinate $z$ along the axes of the cylinder and the azimuth $\phi$ can be assumed; $n$ and $\alpha$ are an integer and real, respectively, parameters of the problem.) In [*Zheligovsky et al.*, 2001], a limited scale separation was studied by considering magnetic modes for wave vectors $\mathbf{q}$, whose components are only 0 or 1/2 (we henceforth call such wave vectors *half-integer*).

Homogenization of elliptic operators was also employed for the study of large-scale perturbations of steady flows (in the absence of magnetic field) [*Dubrulle and Frisch*, 1991] and of magnetohydrodynamic (MHD) steady states [*Zheligovsky*, 2003] (see also [*Zheligovsky*, 2011]). The three problems are similar: in the hydrodynamic stability problem, the evolution of the mean field is generically controlled by the anisotropic kinetic $\alpha$-effect (aka the AKA-effect), or by the eddy viscosity in its absence; in the MHD stability problem the combined $\alpha$-effect or the combined eddy diffusivity act in place of the kinetic ones. (A straightforward modification of the formalism is used to study the large-scale stability of small-scale time-periodic or quasi-periodic hydrodynamic and MHD states.)

A question arises naturally, whether the large-scale instabilities predicted by this technique play a significant role in physics of fluids, or they merely provide a demonstration of magnetic, kinetic and MHD instabilities to perturbations involving a spatial scale much larger than the spatial period of the perturbed state? We will address it numerically and report the results in the next paper in this series.

We focus on the small-scale turbulent motion of incompressible fluid and, in the MHD stability problem, on magnetic field, that are characterized by a certain range of spatial scales within a hierarchy of scales, and study their instability to perturbations of larger scales. By this mechanism the energy contained in a given scale range can be transferred to larger-scale structures. The container is assumed to be large enough for the boundaries not to affect the processes of the considered scale lengths. The flow $\mathbf{V}(\mathbf{x})$ and magnetic field $\mathbf{B}(\mathbf{x})$ (when present) that are subjected to the perturbations are supposed to be $2\pi$-periodic in the three-dimensional space, and we explore linear perturbations of the Bloch type.

In the present paper we prepare the tools for implementation of this numerical investigation: we state the Bloch eigenvalue problems for the three linear stability problems at hand and give an exposition of the mathematical results that facilitate their numerical treatment. The plan of the paper is as follows: In section 2 we recall the governing equations for the linear stability analysis that we intend to perform. Using the biorthogonality of eigenfunctions of the operator of linearization and its adjoint, in section 3 we derive an expression for the gradient of the instability mode growth rate, regarded as a function of the wave vector $\mathbf{q}$. It is used in section 4 to show that for half-integer wave vectors the gradient is zero, i.e., for them the necessary condition for the maximum growth rate is satisfied, provided that the MHD steady state experiencing the perturbation is parity-invariant or the respective eigenvalue of the operator of linearization is real; this result justifies the choice of wave vectors in [*Zheligovsky et al.*, 2001]. In section 5 we review the results of the multiscale stability theory. Our concluding remarks are summarized in the last section.

## 2. The governing equations of the MHD stability problem

We consider linear stability of an MHD state $(\mathbf{V}(\mathbf{x}), \mathbf{B}(\mathbf{x}))$, which, for the sake of simplicity, is supposed to be steady. It satisfies the usual system of equations:

$$\nu\nabla^2\mathbf{V} + \mathbf{V}\times(\nabla\times\mathbf{V}) + (\nabla\times\mathbf{B})\times\mathbf{B} - \nabla P + \mathbf{F} = 0, \tag{1.1}$$

$$\eta\nabla^2\mathbf{B} + \nabla\times(\mathbf{V}\times\mathbf{B}) + \mathbf{J} = 0, \tag{1.2}$$

$$\nabla\cdot\mathbf{V} = \nabla\cdot\mathbf{B} = 0. \tag{1.3}$$

Here $\mathbf{V}$, $\mathbf{B}$ and $P$ are the flow velocity, magnetic field and pressure of the spatial scale under investigation, the source terms $\mathbf{F}$ and $\mathbf{J}$ describe the influence of other scale ranges on the flow $\mathbf{V}$ and magnetic field $\mathbf{B}$, $\nu$ is the molecular viscosity of the fluid and $\eta$ its magnetic molecular diffusivity.

As usual in the linear stability analysis, we study perturbed regimes of the form $(\mathbf{V}(\mathbf{x})+\mathbf{V}'(\mathbf{x})\mathrm{e}^{\lambda t},\ \mathbf{B}(\mathbf{x})+\mathbf{B}'(\mathbf{x})\mathrm{e}^{\lambda t},\ P(x)+P'(x)\mathrm{e}^{\lambda t})$. Assuming an exponential time dependence of the perturbation, we reduce the stability problem to an eigenvalue problem for the perturbation mode $(\mathbf{V}',\mathbf{B}')$:

$$\nu\nabla^2\mathbf{V}'+\mathbf{V}'\times(\nabla\times\mathbf{V})+\mathbf{V}\times(\nabla\times\mathbf{V}')+(\nabla\times\mathbf{B}')\times\mathbf{B}+(\nabla\times\mathbf{B})\times\mathbf{B}'-\nabla P'=\lambda\mathbf{V}', \tag{2.1}$$

$$\eta\nabla^2\mathbf{B}'+\nabla\times(\mathbf{V}'\times\mathbf{B}+\mathbf{V}\times\mathbf{B}')=\lambda\mathbf{B}', \tag{2.2}$$

$$\nabla\cdot\mathbf{V}'=\nabla\cdot\mathbf{B}'=0. \tag{2.3}$$

The pressure perturbation $P'$ satisfies the Poisson equation expressing the solenoidality of the l.h.s. of (2.1).

We study perturbation modes of the Bloch type

$$(\mathbf{V}'(\mathbf{x}),\ \mathbf{B}'(\mathbf{x}))=\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}(\mathbf{v}(\mathbf{x}),\ \mathbf{b}(\mathbf{x})), \tag{3.1}$$

$$P'(\mathbf{x})=\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}p(\mathbf{x}), \tag{3.2}$$

where the fields $(\mathbf{v},\mathbf{b})$ have the same spatial periodicity cell $\mathbb{T}^3=[-\pi,\pi]^3$ as the perturbed steady state $(\mathbf{V},\mathbf{B})$. We seek the wave vector $\mathbf{q}$ for which (for a given $\nu$ and $\eta$) the growth rate of the mode (3.1), $\gamma(\mathbf{q})=\mathrm{Re}\,\lambda(\mathbf{q})$, attains the maximum over all $\mathbf{q}$. It suffices to consider $\mathbf{q}$ in the cube $|q_m|\le 1/2$, since

$$\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}(\mathbf{v},\mathbf{b})=\mathrm{e}^{\mathrm{i}(\mathbf{q}-\mathbf{n})\cdot\mathbf{x}}\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}}(\mathbf{v},\mathbf{b}),$$

and the field $\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}}(\mathbf{v},\mathbf{b})$ has the same spatial periods as the perturbed state $\mathbf{V}(\mathbf{x}),\mathbf{B}(\mathbf{x})$ for any integer-component $\mathbf{n}$.

Let us define the projection of a Bloch field $\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathbf{f}(\mathbf{x})$ onto the subspace of solenoidal vector fields. A field $\mathbf{f}$, $2\pi$-periodic in each $x_k$, admits a Fourier series expansion

$$\mathbf{f}(\mathbf{x})=\sum_{\mathbf{n}}\hat{\mathbf{f}}_{\mathbf{n}}\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}},$$

where summation is over all three-dimensional integer-component vectors $\mathbf{n}$. Solenoidality of the field $\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathbf{f}$ is equivalent to the orthogonality $\hat{\mathbf{f}}_{\mathbf{n}}\cdot(\mathbf{n}+\mathbf{q})=0$ for all wave vectors $\mathbf{n}$. We set

$$\mathscr{P}_{\mathbf{q}}:\mathbf{f}\mapsto\sum_{\mathbf{n}}\left(\hat{\mathbf{f}}_{\mathbf{n}}-\frac{\hat{\mathbf{f}}_{\mathbf{n}}\cdot(\mathbf{n}+\mathbf{q})}{|\mathbf{n}+\mathbf{q}|^2}(\mathbf{n}+\mathbf{q})\right)\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}}, \tag{4.1}$$

well-defined for $\mathbf{q}\ne 0$, and

$$\mathscr{P}_0:\mathbf{f}\mapsto\hat{\mathbf{f}}_0+\sum_{\mathbf{n}\ne 0}\left(\hat{\mathbf{f}}_{\mathbf{n}}-\frac{\hat{\mathbf{f}}_{\mathbf{n}}\cdot\mathbf{n}}{|\mathbf{n}|^2}\mathbf{n}\right)\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}} \tag{4.2}$$

for $\mathbf{q}=0$. Then $\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathscr{P}_{\mathbf{q}}\mathbf{f}$ is the solenoidal part of $\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathbf{f}(\mathbf{x})$ and $\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}(\mathscr{I}-\mathscr{P}_{\mathbf{q}})\mathbf{f}$ is a gradient; here $\mathscr{I}$ denotes the identity operator. It is straightforward to show that

$$\overline{\mathscr{P}_{\mathbf{q}}\mathbf{f}}=\mathscr{P}_{-\mathbf{q}}\overline{\mathbf{f}}, \tag{5}$$

where the bar denotes complex conjugation.

For MHD perturbation modes (3), the problem (2) reduces to the eigenvalue problem for their $\mathbb{T}^3$-periodic parts,

$$\mathscr{M}_{\mathbf{q}}(\mathbf{v},\mathbf{b})=\lambda(\mathbf{q})(\mathbf{v},\mathbf{b}), \tag{6}$$

for the operator

$$\mathscr{M}_{\mathbf{q}} : (\mathbf{v}, \mathbf{b}) \mapsto \Big(\nu \Delta_{\mathbf{q}} \mathbf{v} + \mathscr{P}_{\mathbf{q}}\Big(\mathbf{v} \times (\nabla \times \mathbf{V}) + \mathbf{V} \times (\nabla \times \mathbf{v}) + \mathrm{i}\mathbf{V} \times (\mathbf{q} \times \mathbf{v}) \\ + (\nabla \times \mathbf{b}) \times \mathbf{B} + \mathrm{i}(\mathbf{q} \times \mathbf{b}) \times \mathbf{B} + (\nabla \times \mathbf{B}) \times \mathbf{b}\Big), \\ \eta \Delta_{\mathbf{q}} \mathbf{b} + \nabla \times (\mathbf{v} \times \mathbf{B} + \mathbf{V} \times \mathbf{b}) + \mathrm{i}\mathbf{q} \times (\mathbf{v} \times \mathbf{B} + \mathbf{V} \times \mathbf{b})\Big). \quad (7)$$

Here

$$\Delta_{\mathbf{q}} : \mathbf{v} \mapsto \nabla^2 \mathbf{v} + 2\mathrm{i}(\mathbf{q} \cdot \nabla)\mathbf{v} - |\mathbf{q}|^2 \mathbf{v} \quad (8)$$

is a self-adjoint operator with respect to the scalar product

$$\langle\langle \mathbf{f}_1, \mathbf{f}_2 \rangle\rangle = \left\langle \mathbf{f}_1 \cdot \overline{\mathbf{f}_2} \right\rangle \equiv (2\pi)^{-3} \int_{\mathbb{T}^3} \mathbf{f}_1(\mathbf{x}) \cdot \overline{\mathbf{f}_2(\mathbf{x})} \, \mathrm{d}\mathbf{x}$$

in the functional Lebesgue space $\mathbb{L}_2(\mathbb{T}^3)$ (we use the product for vector fields $\mathbf{f}_i$ in $\mathbb{C}^3$ and $\mathbb{C}^6$). For $\mathbf{q} \neq 0$, $\Delta_{\mathbf{q}}$ is negatively defined. In view of (5), complex conjugation of the equation (6) establishes that the modes (3.1) for opposite $\mathbf{q}$ have the same growth rates (the associated eigenfunctions are complex conjugate), and thus it suffices to search for the maximum growth rate in the parallelepiped

$$\mathbb{Q} = \{\mathbf{q} \mid 0 \leq q_1 \leq 1/2, \quad -1/2 \leq q_2 \leq 1/2, \quad -1/2 \leq q_3 \leq 1/2\}.$$

Since the Lorentz force $(\nabla \times \mathbf{B}) \times \mathbf{B}$ in the Navier–Stokes equation (1.1) is quadratic in magnetic field, for $\mathbf{B} = 0$ the MHD stability problem splits into two independent problems: the hydrodynamic stability problem for the flow perturbation modes $(\mathbf{V}'(\mathbf{x}), 0)$, and the kinematic dynamo problem for magnetic perturbations $(0, \mathbf{B}'(\mathbf{x}))$. We consider all the three stability problems.

## 3. Computation of the gradient of the growth rates

The dominant (i.e., maximum over all wave vectors $\mathbf{q}$) growth rates of stability modes can be computed by applying a quasi-Newton method of the steepest descent type (such as the variable metric method BFGS [*Press et al.*, 1992]). At each step, this requires computing the gradient of the growth rate in components of $\mathbf{q}$. We express it in terms of the respective eigenfunction of the adjoint operator.

We henceforth assume for simplicity that $\lambda$ is of multiplicity one, which is the generic case for $\mathbf{q} \neq 0$; we call such eigenvalues *simple*. (The important non-generic case is $\lambda = 0$ for $\mathbf{q} = 0$, because the kernel of the operator of linearization is three-dimensional in the kinematic dynamo and hydrodynamic stability problems, and six-dimensional in the MHD stability problems. To calculate the gradient in this case, the same approach is applicable, but this problem is algebraically more involved.) We can find its derivatives in $q_m$ using the biorthogonality of the eigenfunctions of a linear operator and its adjoint. Differentiating (6) in $q_m$ yields (if the derivative exists)

$$(\mathscr{M}_{\mathbf{q}} - \lambda)\left(\frac{\partial}{\partial q_m}(\mathbf{v}, \mathbf{b})\right) + (\zeta^{\mathrm{v}}, \zeta^{\mathrm{b}}) = \frac{\partial \lambda}{\partial q_m}(\mathbf{v}, \mathbf{b}), \quad (9)$$

where

$$\zeta^{\mathrm{v}} = 2\nu\left(-q_m\mathbf{v} + \mathrm{i}\,\frac{\partial\mathbf{v}}{\partial x_m}\right) + \mathrm{i}\mathscr{P}_{\mathbf{q}}(\mathbf{V}\times(\mathbf{e}_m\times\mathbf{v}) + (\mathbf{e}_m\times\mathbf{b})\times\mathbf{B}) + \frac{\partial\mathscr{P}_{\mathbf{q}}}{\partial q_m}\Big(\mathbf{v}\times(\nabla\times\mathbf{V})$$
$$+\mathbf{V}\times(\nabla\times\mathbf{v}) + \mathrm{i}\mathbf{V}\times(\mathbf{q}\times\mathbf{v}) + (\nabla\times\mathbf{b})\times\mathbf{B} + \mathrm{i}(\mathbf{q}\times\mathbf{b})\times\mathbf{B} + (\nabla\times\mathbf{B})\times\mathbf{b}\Big),$$
$$\zeta^{\mathrm{b}} = 2\eta\left(-q_m\mathbf{b} + \mathrm{i}\,\frac{\partial\mathbf{b}}{\partial x_m}\right) + \mathrm{i}\mathbf{e}_m\times(\mathbf{v}\times\mathbf{B} + \mathbf{V}\times\mathbf{b}),$$

$\mathbf{e}_m$ are unit vectors of the Cartesian coordinate system and

$$\left(\frac{\partial\mathscr{P}_{\mathbf{q}}}{\partial q_m}\right)\!: \mathbf{f}\mapsto\sum_{\mathbf{n}}\left(2(n_m+q_m)\frac{\hat{\mathbf{f}}_{\mathbf{n}}\cdot(\mathbf{n}+\mathbf{q})}{|\mathbf{n}+\mathbf{q}|^4}(\mathbf{n}+\mathbf{q}) - \frac{\hat{\mathbf{f}}_{\mathbf{n}}\cdot\mathbf{e}_m}{|\mathbf{n}+\mathbf{q}|^2}(\mathbf{n}+\mathbf{q}) - \frac{\hat{\mathbf{f}}_{\mathbf{n}}\cdot(\mathbf{n}+\mathbf{q})}{|\mathbf{n}+\mathbf{q}|^2}\mathbf{e}_m\right)\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}}. \quad (10)$$

The operator $\mathscr{O}^*$ adjoint to $\mathscr{O}$ in $\mathbb{L}_2(\mathbb{T}^3)$ is defined as usual by the identity

$$\langle\!\langle\mathscr{O}(\mathbf{v}_1,\mathbf{b}_1),(\mathbf{v}_2,\mathbf{b}_2)\rangle\!\rangle = \langle\!\langle(\mathbf{v}_1,\mathbf{b}_1),\mathscr{O}^*(\mathbf{v}_2,\mathbf{b}_2)\rangle\!\rangle$$

that holds true for any pair $(\mathbf{v}_1,\mathbf{b}_1),(\mathbf{v}_2,\mathbf{b}_2)$ of three-dimensional $\mathbb{T}^3$-periodic smooth fields $\mathbf{v}_k$ and $\mathbf{b}_k$ (not necessarily solenoidal ones). It is easy to verify that the projection $\mathscr{P}_{\mathbf{q}}$ is self-adjoint, and

$$\mathscr{M}^*_{\mathbf{q}}:(\mathbf{v},\mathbf{b})\mapsto\Big(\nu\Delta_{\mathbf{q}}\mathbf{v} + (\nabla\times\mathbf{V})\times\mathscr{P}_{\mathbf{q}}\mathbf{v} + \nabla\times(\mathscr{P}_{\mathbf{q}}\mathbf{v}\times\mathbf{V}) + \mathrm{i}\mathbf{q}\times(\mathscr{P}_{\mathbf{q}}\mathbf{v}\times\mathbf{V})$$
$$-(\nabla\times\mathbf{b} + \mathrm{i}\mathbf{q}\times\mathbf{b})\times\mathbf{B},$$
$$\eta\Delta_{\mathbf{q}}\mathbf{b} + \nabla\times(\mathbf{B}\times\mathscr{P}_{\mathbf{q}}\mathbf{v}) + \mathrm{i}\mathbf{q}\times(\mathbf{B}\times\mathscr{P}_{\mathbf{q}}\mathbf{v}) - (\nabla\times\mathbf{B})\times\mathscr{P}_{\mathbf{q}}\mathbf{v}$$
$$-\mathbf{V}\times(\nabla\times\mathbf{b} + \mathrm{i}\mathbf{q}\times\mathbf{b})\Big). \quad (11)$$

If $\lambda$ is an eigenvalue of $\mathscr{M}_{\mathbf{q}}$, then $\overline{\lambda}$ is an eigenvalue of the adjoint operator $\mathscr{M}^*_{\mathbf{q}}$. For solving the problem

$$\mathscr{M}^*_{\mathbf{q}}(\mathbf{v}^*,\mathbf{b}^*) = \overline{\lambda}(\mathbf{v}^*,\mathbf{b}^*) \quad (12)$$

efficiently, we employ the projection $\mathscr{P}_{\mathbf{q}}$. After we apply $\mathscr{P}_{\mathbf{q}}$ to the hydrodynamic component of (12), the transformed eigenvalue problem involves $\mathscr{P}_{\mathbf{q}}\mathbf{v}^*$, but not the field $\mathbf{v}^*$ individually (since the projection $\mathscr{P}_{\mathbf{q}}$ and the modified Laplacian $\Delta_{\mathbf{q}}$ commute). In view of the identity

$$\nabla\times\mathbf{b}^* + \mathrm{i}\mathbf{q}\times\mathbf{b}^* = \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\nabla\times(\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathbf{b}^*) = \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\nabla\times(\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathscr{P}_{\mathbf{q}}\mathbf{b}^*) = \nabla\times\mathscr{P}_{\mathbf{q}}\mathbf{b}^* + \mathrm{i}\mathbf{q}\times\mathscr{P}_{\mathbf{q}}\mathbf{b}^*,$$

upon applying $\mathscr{P}_{\mathbf{q}}$ to the magnetic component, the new equations involve $\mathbf{b}^*$ also only as the argument of $\mathscr{P}_{\mathbf{q}}$. This decreases the dimension of the functional subspace, where the solution is sought, and hence solving the modified problem requires less computational resources than solving (12). For $\mathscr{P}_{\mathbf{q}}\mathbf{v}^*$ and $\mathscr{P}_{\mathbf{q}}\mathbf{b}^*$ known, (12) yields

$$(\mathscr{I}-\mathscr{P}_{\mathbf{q}})\mathbf{v}^* = (\overline{\lambda}-\nu\Delta_{\mathbf{q}})^{-1}(\mathscr{I}-\mathscr{P}_{\mathbf{q}})\Big((\nabla\times\mathbf{V})\times\mathscr{P}_{\mathbf{q}}\mathbf{v}^* - (\nabla\times\mathscr{P}_{\mathbf{q}}\mathbf{b}^* + \mathrm{i}\mathbf{q}\times\mathscr{P}_{\mathbf{q}}\mathbf{b}^*)\times\mathbf{B}\Big),$$
$$(\mathscr{I}-\mathscr{P}_{\mathbf{q}})\mathbf{b}^* = -(\overline{\lambda}-\eta\Delta_{\mathbf{q}})^{-1}(\mathscr{I}-\mathscr{P}_{\mathbf{q}})\Big((\nabla\times\mathbf{B})\times\mathscr{P}_{\mathbf{q}}\mathbf{v}^* + \mathbf{V}\times(\nabla\times\mathscr{P}_{\mathbf{q}}\mathbf{b}^* + \mathrm{i}\mathbf{q}\times\mathscr{P}_{\mathbf{q}}\mathbf{b}^*)\Big).$$

The two orthogonal complements are easily obtained in the Fourier space. We normalize the eigenfunction $(\mathbf{v}^*,\mathbf{b}^*)$ by imposing the condition

$$\langle\!\langle(\mathbf{v},\mathbf{b}),(\mathbf{v}^*,\mathbf{b}^*)\rangle\!\rangle = 1. \quad (13)$$

Scalar multiplying now (9) by $(\mathbf{v}^*,\mathbf{b}^*)$ and taking the real part, we find the gradient of the growth rate of the perturbation,

$$\begin{aligned}\frac{\partial\gamma}{\partial q_m} &= \mathrm{Re}\,\langle\langle(\zeta^{\mathrm{v}},\zeta^{\mathrm{b}}),(\mathbf{v}^*,\mathbf{b}^*)\rangle\rangle\\ &= \mathrm{Re}\left\langle\!\!\left\langle -2\nu q_m\mathbf{v}+\frac{\partial\mathscr{P}_{\mathbf{q}}}{\partial q_m}\Big(\mathbf{v}\times(\nabla\times\mathbf{V})+\mathbf{V}\times(\nabla\times\mathbf{v}+\mathrm{i}\mathbf{q}\times\mathbf{v})+(\nabla\times\mathbf{b}+\mathrm{i}\mathbf{q}\times\mathbf{b})\times\mathbf{B}\right.\right.\\ &\quad\left.\left. +\,(\nabla\times\mathbf{B})\times\mathbf{b}\Big),\mathbf{v}^*\right\rangle\!\!\right\rangle - \mathrm{Im}\left\langle\!\!\left\langle 2\nu\frac{\partial\mathbf{v}}{\partial x_m}+\mathscr{P}_{\mathbf{q}}\Big(\mathbf{V}\times(\mathbf{e}_m\times\mathbf{v})+(\mathbf{e}_m\times\mathbf{b})\times\mathbf{B}\Big),\mathbf{v}^*\right\rangle\!\!\right\rangle\\ &\quad -2\eta q_m\mathrm{Re}\langle\langle\mathbf{b},\mathbf{b}^*\rangle\rangle - \mathrm{Im}\left\langle\!\!\left\langle 2\eta\frac{\partial\mathbf{b}}{\partial x_m}+\mathbf{e}_m\times(\mathbf{v}\times\mathbf{B}+\mathbf{V}\times\mathbf{b}),\mathbf{b}^*\right\rangle\!\!\right\rangle. \end{aligned}\tag{14}$$

By construction,

$$\mathscr{M}_{\mathbf{q}}(\mathbf{v},\mathbf{b}) = \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathscr{M}(\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}(\mathbf{v},\mathbf{b})),\tag{15}$$

where $\mathscr{M}$ is the linearization (see (2)) of the system of the MHD equations (1); its explicit form, as well as that of the adjoint operator $\mathscr{M}^*$, can be obtained by setting $\mathbf{q}=0$ in (7) and (11), respectively. In particular,

$$\begin{aligned}\mathscr{M}:(\mathbf{v},\mathbf{b})\mapsto\Big(&\nu\nabla^2\mathbf{v}+\mathscr{P}(\mathbf{v}\times(\nabla\times\mathbf{V})+\mathbf{V}\times(\nabla\times\mathbf{v})+(\nabla\times\mathbf{b})\times\mathbf{B}+(\nabla\times\mathbf{B})\times\mathbf{b}),\\ &\eta\nabla^2\mathbf{b}+\nabla\times(\mathbf{v}\times\mathbf{B}+\mathbf{V}\times\mathbf{b})\Big),\end{aligned}\tag{16.1}$$

where $\mathscr{P}=\mathscr{P}_0$ is the projection into the space of solenoidal fields,

$$\mathscr{P}:\mathbf{f}\mapsto\mathbf{f}-\nabla(\nabla^{-2}(\nabla\cdot\mathbf{f})),\tag{16.2}$$

and $\nabla^{-2}$ denotes the inverse Laplacian. A similar relation holds for the adjoint operators:

$$\mathscr{M}^*_{\mathbf{q}}(\mathbf{v},\mathbf{b}) = \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathscr{M}^*(\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}(\mathbf{v},\mathbf{b})).\tag{17}$$

These identities are used both for the analysis of the problem (see the next section) and in the numerical work.

The operators encountered in the hydrodynamic stability problem are obtained from (7) and (11) for $\mathbf{B}=\mathbf{b}=0$,

$$\mathscr{H}_{\mathbf{q}}:\mathbf{v}\mapsto\nu\Delta_{\mathbf{q}}\mathbf{v}+\mathscr{P}_{\mathbf{q}}\Big(\mathbf{v}\times(\nabla\times\mathbf{V})+\mathbf{V}\times(\nabla\times\mathbf{v})+\mathrm{i}\mathbf{V}\times(\mathbf{q}\times\mathbf{v})\Big),\tag{18.1}$$

$$\mathscr{H}^*_{\mathbf{q}}:\mathbf{v}\mapsto\nu\Delta_{\mathbf{q}}\mathbf{v}+(\nabla\times\mathbf{V})\times\mathscr{P}_{\mathbf{q}}\mathbf{v}+\nabla\times(\mathscr{P}_{\mathbf{q}}\mathbf{v}\times\mathbf{V})+\mathrm{i}\mathbf{q}\times(\mathscr{P}_{\mathbf{q}}\mathbf{v}\times\mathbf{V})\tag{18.2}$$

and in the kinematic dynamo problem for $\mathbf{B}=\mathbf{v}=0$,

$$\mathscr{D}_{\mathbf{q}}:\mathbf{b}\mapsto\eta\Delta_{\mathbf{q}}\mathbf{b}+\nabla\times(\mathbf{V}\times\mathbf{b})+\mathrm{i}\mathbf{q}\times(\mathbf{V}\times\mathbf{b}),\tag{19.1}$$

$$\mathscr{D}^*_{\mathbf{q}}:\mathbf{b}\mapsto\eta\Delta_{\mathbf{q}}\mathbf{b}-\mathbf{V}\times(\nabla\times\mathbf{b}+\mathrm{i}\mathbf{q}\times\mathbf{b}).\tag{19.2}$$

We denote by $\gamma^{\mathbf{b}}$, $\gamma^{\mathbf{v}}$ and $\gamma^{\mathbf{bv}}$ the maximum, over the wave vectors $\mathbf{q}$, growth rates of the generated magnetic field and perturbations in the kinematic dynamo, hydrodynamic and MHD linear stability problems, respectively.

For computing $\mathscr{M}_{\mathbf{q}}(\mathbf{v},\mathbf{b})$, the code for computation of the linearization $\mathscr{M}(\mathbf{v},\mathbf{b})$ (16) can be used upon transforming, following (15), the previously integer wave numbers $n_k\to n_k+q_k$ when computing all spatial derivatives except in $\nabla\times\mathbf{V}$ and $\nabla\times\mathbf{B}$. This way we avoid programming the additional (involving $\mathbf{q}$) terms in (7) and spending the processor time to compute them separately. The same approach can be used, in view of (17), for computing the results of the action of the adjoint operator $\mathscr{M}^*_{\mathbf{q}}$ (11): it suffices to use the code for computation of $\mathscr{M}^*$, adjoint to $\mathscr{M}$, for the transformed wave numbers. (Of course, this remark also applies to computations for the two stability problems for an amagnetic steady state.) Similarly, the solenoidality of the fields constituting the mode (3.1) in terms of the coefficients of the Fourier series

$$\mathbf{v} = \sum_{\mathbf{n}} \hat{\mathbf{v}}_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}}, \quad \mathbf{b} = \sum_{\mathbf{n}} \hat{\mathbf{b}}_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}}$$

is equivalent to the orthogonality $\hat{\mathbf{v}}_{\mathbf{n}} \cdot (\mathbf{n}+\mathbf{q}) = \hat{\mathbf{b}}_{\mathbf{n}} \cdot (\mathbf{n}+\mathbf{q}) = 0$ for all wave vectors $\mathbf{n}$.

The magnetic induction operator $\mathscr{D} : \mathbf{b} \mapsto \eta\nabla^2\mathbf{b} + \nabla\times(\mathbf{V}\times\mathbf{b})$ arising in the kinematic dynamo problem has a well-known property:

$$\nabla\times(\mathscr{D}^*\mathbf{b}) = \mathscr{D}^-(\nabla\times\mathbf{b})$$

(see, e.g., [*Rasskazov et al.*, 2018]), where $\mathscr{D}^- : \mathbf{b} \mapsto \eta\nabla^2\mathbf{b} - \nabla\times(\mathbf{V}\times\mathbf{b})$ denotes the magnetic induction operator for the reverse flow $-\mathbf{V}$. Consequently, the eigenvalue problem

$$\mathscr{D}^*_{\mathbf{q}}\mathbf{b}^* = \overline{\lambda}\mathbf{b}^* \tag{20}$$

can be solved using the code for the kinematic dynamo problem for the reverse flow,

$$\eta\Delta_{\mathbf{q}}\mathbf{c}^* - \nabla\times(\mathbf{V}\times\mathbf{c}^*) - \mathrm{i}\mathbf{q}\times(\mathbf{V}\times\mathbf{c}^*) = \overline{\lambda}\mathbf{c}^*,$$

(upon transforming the wave numbers $n_k \to n_k + q_k$, as discussed in the previous paragraph). Substituting its solution $\mathbf{c}^* = \nabla\times\mathbf{b}^* + \mathrm{i}\mathbf{q}\times\mathbf{b}^*$ into (20) yields an elliptic equation, easily solvable in the Fourier space and furnishing $\mathbf{b}^* = (\eta\Delta_{\mathbf{q}} - \overline{\lambda})^{-1}(\mathbf{V}\times\mathbf{c}^*)$.

The ability to compute the gradient (14) suggests that, in order to find the maximum growth rate, we might seek solutions to the system of equations

$$\partial\gamma/\partial q_m = 0. \tag{21}$$

However, this is not optimal for several reasons: Solving numerically a nonlinear system of equations can be problematic [*Press et al.*, 1992]. The gradient can have singularities near bifurcation points at a branch of eigenvectors of a linear operator (in particular, when the branch emerges or disappears). Furthermore, in different regions of the $\mathbf{q}$ space the locally maximum growth rate can be attained in distinct branches; it is continuous, but not differentiable at the borders between the regions.

In order to check, whether a wave vector $\mathbf{q}$, for which (21) is satisfied, is associated with a locally maximum growth rate (for instance, when following branches of eigenvalues for half-integer wave vectors), we can analyze the eigenvalues of the Hessian $\|\partial^2\mathrm{Re}\lambda/\partial q_m\partial q_n\|$. For a simple eigenvalue, the second derivatives comprising this matrix are obtained by differentiating (14) in $q_n$. The resulting expression involves the derivatives $\partial(\mathbf{v},\mathbf{b})/\partial q_n$ and $\partial(\mathbf{v}^*,\mathbf{b}^*)/\partial q_n$. These vector fields satisfy (9) and the equation obtained by differentiating (14) in $q_m$,

$$(\mathscr{M}^*_{\mathbf{q}} - \overline{\lambda})\left(\frac{\partial}{\partial q_m}(\mathbf{v}^*,\mathbf{b}^*)\right) + (\zeta^{\mathrm{v}*},\zeta^{\mathrm{b}*}) = \frac{\partial\overline{\lambda}}{\partial q_m}(\mathbf{v}^*,\mathbf{b}^*), \tag{22}$$

where

$$\zeta^{\mathrm{v}*} = 2\nu\left(-q_m\mathbf{v}^* + \mathrm{i}\frac{\partial\mathbf{v}^*}{\partial x_m}\right) + (\nabla\times\mathbf{V})\times\frac{\partial\mathscr{P}_{\mathbf{q}}}{\partial q_m}\mathbf{v}^* + \nabla\times\left(\frac{\partial\mathscr{P}_{\mathbf{q}}}{\partial q_m}\mathbf{v}^*\times\mathbf{V}\right)$$
$$+\,\mathrm{i}\mathbf{q}\times\left(\frac{\partial\mathscr{P}_{\mathbf{q}}}{\partial q_m}\mathbf{v}^*\times\mathbf{V}\right) + \mathrm{i}\mathbf{e}_m\times(\mathscr{P}_{\mathbf{q}}\mathbf{v}^*\times\mathbf{V}) - \mathrm{i}(\mathbf{e}_m\times\mathbf{b}^*)\times\mathbf{B},$$
$$\zeta^{\mathrm{b}*} = 2\eta\left(-q_m\mathbf{b}^* + \mathrm{i}\frac{\partial\mathbf{b}^*}{\partial x_m}\right) + \nabla\times\left(\mathbf{B}\times\frac{\partial\mathscr{P}_{\mathbf{q}}}{\partial q_m}\mathbf{v}^*\right) + \mathrm{i}\mathbf{q}\times\left(\mathbf{B}\times\frac{\partial\mathscr{P}_{\mathbf{q}}}{\partial q_m}\mathbf{v}^*\right) + \mathrm{i}\mathbf{e}_m\times(\mathbf{B}\times\mathscr{P}_{\mathbf{q}}\mathbf{v}^*)$$
$$-\,(\nabla\times\mathbf{B})\times\frac{\partial\mathscr{P}_{\mathbf{q}}}{\partial q_m}\mathbf{v}^* - \mathrm{i}\mathbf{V}\times(\mathbf{e}_m\times\mathbf{b}^*).$$

If the eigenvalue $\lambda$ is simple, the solvability conditions for the two equations, (9) and (22), reduce to

$$\frac{\partial\lambda}{\partial q_m} = \langle\langle(\zeta^{\mathrm{v}}, \zeta^{\mathrm{b}}), (\mathbf{v}^*, \mathbf{b}^*)\rangle\rangle.$$

Solving these equations, we can determine the derivatives $\partial(\mathbf{v}, \mathbf{b})/\partial q_m$ and $\partial(\mathbf{v}^*, \mathbf{b}^*)/\partial q_m$ up to arbitrary additive terms proportional to the eigenvector of $\mathscr{M}_{\mathbf{q}}$ ($\mathscr{M}^*_{\mathbf{q}}$, respectively) associated with the eigenvalues $\lambda$ ($\overline{\lambda}$, respectively). These terms are chosen in such a way that the normalization condition

$$\left\langle\!\!\left\langle (\mathbf{v}, \mathbf{b}), \frac{\partial}{\partial q_m}(\mathbf{v}^*, \mathbf{b}^*) \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \frac{\partial}{\partial q_m}(\mathbf{v}, \mathbf{b}), (\mathbf{v}^*, \mathbf{b}^*) \right\rangle\!\!\right\rangle = 0$$

(obtained by differentiating (13) in $q_m$) is satisfied.

For $\mathbf{q} = 0$, the kernel of the linearization has dimension at least 3 for the problems of hydrodynamic stability and kinematic magnetic dynamo, and at least 6 for the full MHD linear stability problem. Consequently, for neutral stability modes for $\mathbf{q} = 0$, the approach outlined above is inapplicable, but the multiscale analysis (see section 5) establishes when zero is a locally maximum growth rate. For a generic steady flow or MHD state, the $\alpha$-effect acting on large-scale fields (Bloch modes for small $\mathbf{q}$) is not offset by the action of diffusion of the same order $\mathrm{O}(|\mathbf{q}|)$ of smallness; consequently, instability to large-scale perturbations persists for all diffusivities $\eta$ and/or viscosities $\nu$, i.e., for any sufficiently small $|\mathbf{q}|$. In the absence of the $\alpha$-effect (e.g., for parity-invariant flows or MHD states), the main mechanism for development of large-scale instability is negative eddy diffusivity (or eddy viscosity in the hydrodynamic problem), and zero is the locally maximum instability growth rate whenever eddy diffusivity is not negative.

## 4. Stationary half-integer wave vectors

Here we show that half-integer wave vectors $\mathbf{q}$ satisfy (21) provided either ($i$) the eigenvalue $\lambda$ is real, or ($ii$) the perturbed state is comprised of parity-invariant fields $\mathbf{V}$ and $\mathbf{B}$, i.e.,

$$\mathbf{V}(\mathbf{x}) = -\mathbf{V}(-\mathbf{x}), \qquad \mathbf{B}(\mathbf{x}) = -\mathbf{B}(-\mathbf{x}) \tag{23}$$

provided the center of symmetry coincides with the origin of the coordinate system. We consider a simple eigenvalue $\lambda$ in order to be able to use the expression (14) for the derivative in the l.h.s. of (21). In other words, under the stated conditions, half-integer wave vectors are stationary points of the growth rates $\gamma(\mathbf{q})$. We consider the two cases following [*Zheligovsky and Chertovskih*, 2020], where this was shown in the context of the kinematic dynamo problem. The existence of the derivatives is assumed.

*(i) The eigenvalue $\lambda$ is real.* Suppose $\mathbf{q} = 0$ (and hence all terms proportional to $q_m$ vanish in the r.h.s. of (14)). Then the eigenfunctions $(\mathbf{v}, \mathbf{b})$ and $(\mathbf{v}^*, \mathbf{b}^*)$ of $\mathscr{M}_{\mathbf{q}}$ and $\mathscr{M}^*_{\mathbf{q}}$ are real-valued (otherwise, their real and imaginary parts belong to the respective invariant subspace) implying that all terms in the r.h.s. of (14) that are imaginary parts vanish. For $\mathbf{q} = 0$, the coefficients in the l.h.s. of the Fourier series (10) are odd in $\mathbf{n}$; thus, the operator $\partial\mathscr{P}_{\mathbf{q}}/\partial q_m$ maps real vector fields to imaginary fields. This establishes (21) for $\mathbf{q} = 0$. For other half-integer $\mathbf{q}$, the proof is more technical, but it is based on the same ideas.

Complex conjugation of (6) with the use of (15) demonstrates that

$$\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathscr{M}(\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathrm{e}^{-2\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\overline{(\mathbf{v}, \mathbf{b})}) = \overline{\lambda(\mathbf{q})}\,\mathrm{e}^{-2\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\overline{(\mathbf{v}, \mathbf{b})},$$

i.e., the $\mathbb{T}^3$-periodic vector field $\mathrm{e}^{-2\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\overline{(\mathbf{v}, \mathbf{b})}$ is also an eigenfunction of $\mathscr{M}_{\mathbf{q}}$ associated with the eigenvalue $\overline{\lambda}$. Consequently, for real $\lambda$, the fields $\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}(\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\overline{(\mathbf{v}, \mathbf{b})} \pm \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}(\mathbf{v}, \mathbf{b}))$ (at least one of which is non-zero) also belong to the same eigenspace. Hence, a solution to (6) takes the form

$$(\mathbf{v}, \mathbf{b}) = \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathbf{g}, \tag{24.1}$$

where $\mathbf{g}(\mathbf{x})$ is real-valued. (Not surprisingly: by (3.1), $e^{i\mathbf{q}\cdot\mathbf{x}}(\mathbf{v},\mathbf{b})=(\mathbf{V}',\mathbf{B}')$ is an MHD perturbation mode satisfying the eigenvalue problem (2) for the linearization $\mathscr{M}$ (16) of the system of equations of magnetohydrodynamics (1); this operator does not involve complex coefficients, and its real eigenfunctions are associated with real eigenvalues $\lambda$. A similar comment pertains to the eigenfunction of the adjoint operator $\mathscr{M}^*$.) In view of the identity (17), by a similar argument, the eigenfunction of $\mathscr{M}^*_{\mathbf{q}}$ associated with the eigenvalue $\lambda$ can be expressed as

$$(\mathbf{v}^*,\mathbf{b}^*)=e^{-i\mathbf{q}\cdot\mathbf{x}}\mathbf{g}^*, \tag{24.2}$$

where $\mathbf{g}^*$ is real-valued (this is compatible with the normalization (13)).

On substituting the eigenfunctions (24) into (14), we find that the two terms involving the factor $\nu$ cancel out (or they both vanish, if $q_m=0$), as well as the two terms involving the factor $\eta$. None of the remaining terms gives a non-zero contribution. In particular, expanding a vector field $\mathbf{f}$ in the Fourier series reveals that if $e^{i\mathbf{q}\cdot\mathbf{x}}\mathbf{f}$ is real, then $e^{i\mathbf{q}\cdot\mathbf{x}}\mathscr{P}_{\mathbf{q}}\mathbf{f}$ is real and $e^{i\mathbf{q}\cdot\mathbf{x}}(\partial\mathscr{P}_{\mathbf{q}}/\partial q_m)\mathbf{f}$ is imaginary; consequently, all terms in (14) involving $\mathscr{P}_{\mathbf{q}}$ or $\partial\mathscr{P}_{\mathbf{q}}/\partial q_m$ vanish. Thus, under our assumptions on $\mathbf{q}$ and $\lambda$, (21) is satisfied.

*(ii) The fields* $\mathbf{V}$ *and* $\mathbf{B}$ *are parity-invariant.* The eigenvalue problem (6) for the operator (15) can now be recast at the point $-\mathbf{x}$ as

$$e^{i\mathbf{q}\cdot\mathbf{x}}\mathscr{M}(e^{-i\mathbf{q}\cdot\mathbf{x}}(\mathbf{v}(-\mathbf{x}),\mathbf{b}(-\mathbf{x})))=\lambda(\mathbf{v}(-\mathbf{x}),\mathbf{b}(-\mathbf{x}))$$

$$\Rightarrow\quad e^{-i\mathbf{q}\cdot\mathbf{x}}\mathscr{M}(e^{i\mathbf{q}\cdot\mathbf{x}}e^{-2i\mathbf{q}\cdot\mathbf{x}}(\mathbf{v}(-\mathbf{x}),\mathbf{b}(-\mathbf{x})))=\lambda\, e^{-2i\mathbf{q}\cdot\mathbf{x}}(\mathbf{v}(-\mathbf{x}),\mathbf{b}(-\mathbf{x})),$$

i.e., $e^{-2i\mathbf{q}\cdot\mathbf{x}}(\mathbf{v}(-\mathbf{x}),\mathbf{b}(-\mathbf{x}))$ is also a $\mathbb{T}^3$-periodic eigenfunction of the operator $\mathscr{M}_{\mathbf{q}}$ associated with the eigenvalue $\lambda$. Since the fields $e^{-i\mathbf{q}\cdot\mathbf{x}}\left(e^{i\mathbf{q}\cdot\mathbf{x}}(\mathbf{v}(\mathbf{x}),\mathbf{b}(\mathbf{x}))\pm e^{-i\mathbf{q}\cdot\mathbf{x}}(\mathbf{v}(-\mathbf{x}),\mathbf{b}(-\mathbf{x}))\right)$ belong to this eigensubspace (one-dimensional by our assumption), the eigenfunction under consideration takes the form (24.1), where $\mathbf{g}$ is parity-invariant ($\mathbf{g}(\mathbf{x})=-\mathbf{g}(-\mathbf{x})$) or parity-antiinvariant ($\mathbf{g}(\mathbf{x})=\mathbf{g}(-\mathbf{x})$); only one of the two possibilities realizes since $\lambda$ is a simple eigenvalue. In terms of the Fourier coefficients $\hat{\mathbf{v}}_{\mathbf{n}}$ and $\hat{\mathbf{b}}_{\mathbf{n}}$ of $\mathbf{v}$ and $\mathbf{b}$, respectively, this property can be expressed as $\hat{\mathbf{v}}_{\mathbf{n}}=-\hat{\mathbf{v}}_{-2\mathbf{q}-\mathbf{n}}$, $\hat{\mathbf{b}}_{\mathbf{n}}=-\hat{\mathbf{b}}_{-2\mathbf{q}-\mathbf{n}}$ for parity-invariant fields $(\mathbf{v},\mathbf{b})$, and $\hat{\mathbf{v}}_{\mathbf{n}}=\hat{\mathbf{v}}_{-2\mathbf{q}-\mathbf{n}}$, $\hat{\mathbf{b}}_{\mathbf{n}}=\hat{\mathbf{b}}_{-2\mathbf{q}-\mathbf{n}}$ for parity-antiinvariant ones. By a similar argument, the eigenfunction of the adjoint operator $\mathscr{M}^*_{\mathbf{q}}$ (17) associated with the eigenvalue $\overline{\lambda}$ has the same structure (24.2), where $\mathbf{g}^*$ is also parity-invariant or antiinvariant. The parity of $\mathbf{g}$ and $\mathbf{g}^*$ is the same, because otherwise $\langle\langle(\mathbf{v},\mathbf{b}),(\mathbf{v}^*,\mathbf{b}^*)\rangle\rangle=0$ which is incompatible with the normalization (13) stemming from the biorthogonality of the bases of eigenfunctions of $\mathscr{M}_{\mathbf{q}}$ and $\mathscr{M}^*_{\mathbf{q}}$ in $\mathbb{L}_2(\mathbb{T}^3)$. (Actually, for a parity-invariant state $(\mathbf{V},\mathbf{B})$, parity-invariant and antiinvariant vector fields constitute invariant subspaces of the linearization $\mathscr{M}$ and its adjoint, implying that their eigenfunctions $\mathbf{g}=e^{i\mathbf{q}\cdot\mathbf{x}}(\mathbf{v},\mathbf{b})$ and $\mathbf{g}^*=e^{i\mathbf{q}\cdot\mathbf{x}}(\mathbf{v}^*,\mathbf{b}^*)$ have the same parity.)

On substituting the expressions (24) into (14), we find that the two terms involving the factor $\nu$ cancel out, as well as the two terms involving the factor $\eta$ (or they are zero if $q_m=0$). By (4.1) and (10), $e^{i\mathbf{q}\cdot\mathbf{x}}\mathscr{P}_{\mathbf{q}}\mathbf{f}$ preserves parity invariance and antiinvariance of $e^{i\mathbf{q}\cdot\mathbf{x}}\mathbf{f}$, and $e^{i\mathbf{q}\cdot\mathbf{x}}(\partial\mathscr{P}_{\mathbf{q}}/\partial q_m)\mathbf{f}$ swaps them. Therefore, all remaining terms in (14) give rise to volume integrals of parity-antiinvariant fields, which are zero. Thus, in the case *(ii)* (21) is also satisfied.

The demonstration is not affected by assuming $\mathbf{B}=0$, i.e., it holds for the hydrodynamic linear stability problem and the kinematic dynamo problem.

As a side remark, we note that, for a parity-invariant steady state $(\mathbf{V},\mathbf{B})$, if $\overline{(\mathbf{v}(\mathbf{x}),\mathbf{b}(\mathbf{x}))}$ is an eigenfunction of the operator $\mathscr{M}_{\mathbf{q}}$ associated with an eigenvalue $\lambda$, the field $\overline{(\mathbf{v}(-\mathbf{x}),\mathbf{b}(-\mathbf{x}))}$ is an eigenfunction of $\mathscr{M}_{\mathbf{q}}$ associated with the eigenvalue $\overline{\lambda}$ (it is straightforward to verify this considering (6) at the point $-\mathbf{x}$ and using the identity $\mathscr{P}_{\mathbf{q}}(\overline{\mathbf{f}(-\mathbf{x})})=(\overline{\mathscr{P}_{\mathbf{q}}\mathbf{f}})(-\mathbf{x})$).

## 5. The formalism of the multiscale stability theory

We outline here the main ideas of the derivation of the combined MHD $\alpha$-effect and diffusivity tensors in the multiscale linear stability theory for MHD steady states residing in the entire space [*Zheligovsky*, 2003] and tune the results for the purposes of the present investigation. Let us stress, that we inspect exclusively the results that are obtained from the first principles by asymptotic methods for systems, where a significant scale separation is present; we are not interested here in the rich variety of results of the mean-field electrodynamics relying on additional assumptions, such as the first-order smoothing approximation (also referred to as the second-order correlation approximation), see, e.g., [*Krause and Rädler*, 1980] and the reviews [*Brandenburg and Subramanian*, 2005; *Brandenburg et al.*, 2012].

A comment on the kernel of the operator $\mathscr{M}$ is in order. It consists of the so-called neutral modes satisfying the equations

$$\nu\nabla^2\mathbf{V}'+\mathbf{V}'\times(\nabla\times\mathbf{V})+\mathbf{V}\times(\nabla\times\mathbf{V}')+(\nabla\times\mathbf{B}')\times\mathbf{B}+(\nabla\times\mathbf{B})\times\mathbf{B}'-\nabla P'=0, \tag{25.1}$$

$$\eta\nabla^2\mathbf{B}'+\nabla\times(\mathbf{V}'\times\mathbf{B}+\mathbf{V}\times\mathbf{B}')=0, \tag{25.2}$$

$$\nabla\cdot\mathbf{V}'=\nabla\cdot\mathbf{B}'=0. \tag{25.3}$$

Let us denote by $\widetilde{\mathscr{M}}$ the restriction of $\mathscr{M}$ on the subspace of $\mathbb{T}^3$-periodic six-dimensional vector fields $(\mathbf{v},\mathbf{b})$ such that both components, $\mathbf{v}$ and $\mathbf{b}$, are three-dimensional solenoidal fields. It is straightforward to show that constant six-dimensional vectors belong to the kernel of its adjoint $\widetilde{\mathscr{M}}^*=\mathscr{P}\mathscr{M}^*$ (here $\mathscr{P}$ is meant to be applied to both components of $\mathscr{M}^*_{\mathbf{q}}$ (11) for $\mathbf{q}=0$). This has two important corollaries (see [*Zheligovsky*, 2011]). First, $\langle\mathbf{f}^{\mathbf{v}}\rangle=\left\langle\mathbf{f}^{\mathbf{b}}\right\rangle=0$ is generically the solvability condition for the equation $\mathscr{M}(\mathbf{v},\mathbf{b})=(\mathbf{f}^{\mathbf{v}},\mathbf{f}^{\mathbf{b}})$ for solenoidal $\mathbf{f}^{\mathbf{v}}$ and $\mathbf{f}^{\mathbf{b}}$, when a solution is supposed to have solenoidal hydrodynamic and magnetic components $\mathbf{v}$ and $\mathbf{b}$ (this condition follows from the Fredholm alternative theorem). Second, the kernel of $\mathscr{M}$ involves six small-scale neutral modes $(\mathbf{V}',\mathbf{B}',P')$, which we denote $\mathbf{S}^{\mathbf{v}}_k(\mathbf{x})=(\mathbf{S}^{\mathbf{vv}}_k(\mathbf{x}),\mathbf{S}^{\mathbf{vb}}_k(\mathbf{x}),S^{\mathbf{v}p}_k(\mathbf{x}))$ and $\mathbf{S}^{\mathbf{b}}_k(\mathbf{x})=(\mathbf{S}^{\mathbf{bv}}_k(\mathbf{x}),\mathbf{S}^{\mathbf{bb}}_k(\mathbf{x}),S^{\mathbf{b}p}_k(\mathbf{x}))$ for $k=1,2,3$; they are normalized by the conditions

$$\left\langle\mathbf{S}^{\mathbf{v}}_k\right\rangle=(\mathbf{e}_k,0,0),\qquad\left\langle\mathbf{S}^{\mathbf{b}}_k\right\rangle=(0,\mathbf{e}_k,0)\ \text{ for }\ 1\le k\le 3$$

and have solenoidal components $\mathbf{S}^{\mathbf{vv}}_k,\mathbf{S}^{\mathbf{vb}}_k,\mathbf{S}^{\mathbf{bv}}_k,\mathbf{S}^{\mathbf{bb}}_k$. Generically, these neutral modes span the kernel of $\mathscr{M}$.

Assuming the dependence of large-scale stability modes on the two spatial variables, the fast (on which the perturbed steady fields $\mathbf{V}(\mathbf{x}),\mathbf{B}(\mathbf{x})$ depend) and slow one, we consider the limit $|\mathbf{q}|=\varepsilon\to 0$ and expand a mode and the associated eigenvalue in the power series

$$\mathbf{V}'=\sum_{n=0}^{\infty}\mathbf{V}'_n(\mathbf{x},\mathbf{X})\varepsilon^n,\qquad\mathbf{B}'=\sum_{n=0}^{\infty}\mathbf{B}'_n(\mathbf{x},\mathbf{X})\varepsilon^n,\qquad P'=\sum_{n=0}^{\infty}P'_n(\mathbf{x},\mathbf{X})\varepsilon^n,\qquad\lambda=\sum_{n=0}^{\infty}\lambda_n\varepsilon^n. \tag{26}$$

Substituting the series into (2) yields a hierarchy of equations of the form

$$\mathscr{M}(\mathbf{V}'_n,\mathbf{B}'_n)=\mathbf{F}_n(\mathbf{V}'_{n-1},\mathbf{B}'_{n-1},\lambda_{n-1},P'_{n-1},\dots,\mathbf{V}'_0,\mathbf{B}'_0,\lambda_0,P'_0). \tag{27}$$

The first equation in the hierarchy is $\mathscr{M}(\mathbf{V}'_0,\mathbf{B}'_0)=\lambda_0(\mathbf{V}'_0,\mathbf{B}'_0)$. We choose $\lambda_0=0$, since, when $\mathrm{Re}\,\lambda_0\neq 0$, a small-$\varepsilon$ perturbation does not change the sign of the real part and thus the large-scale mode has the same stability properties as the unperturbed small-scale mode $(\mathbf{V}'_0,\mathbf{B}'_0)$. Thus, generically we find

$$(\mathbf{V}'_0,\mathbf{B}'_0,P'_0)=\sum_{k=1}^{3}\left(\langle\mathbf{V}'_0\rangle_k\mathbf{S}^{\mathbf{v}}_k(\mathbf{x})+\langle\mathbf{B}'_0\rangle_k\mathbf{S}^{\mathbf{b}}_k(\mathbf{x})\right) \tag{28}$$

(here $\langle\cdot\rangle_k$ denotes the $k$th component of the averaged vector; the subscripts enumerate scalar elements of matrices and vectors; when other indices are present, they are shown after the comma).

The solenoidality of the flow and magnetic field perturbations (2.3) implies

$$\nabla_{\mathbf{X}}\cdot\langle\mathbf{V}'_n\rangle=0,\qquad \nabla_{\mathbf{x}}\cdot\mathbf{V}'_n+\nabla_{\mathbf{X}}\cdot\mathbf{V}'_{n-1}=0, \tag{29.1}$$

$$\nabla_{\mathbf{X}}\cdot\langle\mathbf{B}'_n\rangle=0,\qquad \nabla_{\mathbf{x}}\cdot\mathbf{B}'_n+\nabla_{\mathbf{X}}\cdot\mathbf{B}'_{n-1}=0 \tag{29.2}$$

for all $n\geq 0$ (we assume $\mathbf{V}'_n=\mathbf{B}'_n=0$ for $n<0$), where $\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{X}}$ denote gradients in the fast and slow spatial variables, respectively.

### 5.1. *The combined MHD α-effect*

It is possible to solve the hierarchy of equations (27) at all orders. The solvability condition yields successively equations for the averages $\langle\mathbf{V}'_n\rangle$ and $\langle\mathbf{B}'_n\rangle$, and the terms $\lambda_n$. In view of (29), it reduces for $n=1$ to the relations

$$\sum_{k=1}^{3}\Big(\mathbf{A}^{\mathbf{vv}}_k\,\nabla_{\mathbf{X}}\langle\mathbf{V}'_0\rangle_k+\mathbf{A}^{\mathbf{bv}}_k\,\nabla_{\mathbf{X}}\langle\mathbf{B}'_0\rangle_k\Big)-\nabla_{\mathbf{X}}P^*_0=\lambda_1\langle\mathbf{V}'_0\rangle, \tag{30.1}$$

$$\nabla_{\mathbf{X}}\times\sum_{k=1}^{3}\Big(\mathbf{A}^{\mathbf{vb}}_k\langle\mathbf{V}'_0\rangle_k+\mathbf{A}^{\mathbf{bb}}_k\langle\mathbf{B}'_0\rangle_k\Big)=\lambda_1\langle\mathbf{B}'_0\rangle. \tag{30.2}$$

Here $\mathbf{A}^{\mathbf{vv}}_k$ and $\mathbf{A}^{\mathbf{bv}}_k$ are symmetric $3\times 3$ matrices with the entries

$$A^{\mathbf{vv}}_{k,j'j}=\Big\langle -V_{j'}S^{\mathbf{vv}}_{k,j}-V_jS^{\mathbf{vv}}_{k,j'}+B_{j'}S^{\mathbf{vb}}_{k,j}+B_jS^{\mathbf{vb}}_{k,j'}\Big\rangle, \tag{31.1}$$

$$A^{\mathbf{bv}}_{k,j'j}=\Big\langle -V_{j'}S^{\mathbf{bv}}_{k,j}-V_jS^{\mathbf{bv}}_{k,j'}+B_{j'}S^{\mathbf{bb}}_{k,j}+B_jS^{\mathbf{bb}}_{k,j'}\Big\rangle, \tag{31.2}$$

and $\mathbf{A}^{\mathbf{vb}}_k$ and $\mathbf{A}^{\mathbf{bb}}_k$ are three-dimensional vectors:

$$\mathbf{A}^{\mathbf{vb}}_k=\Big\langle\mathbf{V}\times\mathbf{S}^{\mathbf{vb}}_k-\mathbf{B}\times\mathbf{S}^{\mathbf{vv}}_k\Big\rangle,\quad \mathbf{A}^{\mathbf{bb}}_k=\Big\langle\mathbf{V}\times\mathbf{S}^{\mathbf{bb}}_k-\mathbf{B}\times\mathbf{S}^{\mathbf{bv}}_k\Big\rangle. \tag{31.3}$$

The first-order partial differential operator in the l.h.s. of (30), acting on the vector field $(\langle\mathbf{V}'_0\rangle,\langle\mathbf{B}'_0\rangle)$, is called the *operator of the combined MHD α–effect*.

We denote by $\gamma^{\mathbf{b}}_\alpha$, $\gamma^{\mathbf{v}}_\alpha$ and $\gamma^{\mathbf{bv}}_\alpha$ the maximum slow-time growth rates of the generated magnetic field and perturbations due to the action of the $\alpha$-effects in the kinematic dynamo, hydrodynamic and MHD linear stability problems, respectively. These values are defined as $\max_{\mathbf{q}}\mathrm{Re}\,\lambda_1$, where the coefficient $\lambda_1$ is the leading term in the expansion (26) of the eigenvalue $\lambda$.

The eigenvalue problem (30) can be stated for different boundary conditions for the mean perturbation $(\mathbf{V}',\mathbf{B}')$. As this is often done in the literature, we investigate here the simplest for analysis case of the mean perturbation, space-periodic in the slow variables (whose periodicity may be incompatible with that of the state $(\mathbf{V},\mathbf{B})$ subjected to perturbation). Such a perturbation mode takes the Bloch form (3), and the mean perturbation is then a plain Fourier harmonics

$$\langle(\mathbf{V}'_0,\mathbf{B}'_0)\rangle=(\mathbf{C}_{\mathbf{v}},\mathbf{C}_{\mathbf{b}})\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}, \tag{32}$$

where $\mathbf{C}_{\mathbf{v}}$ and $\mathbf{C}_{\mathbf{b}}$ are constant three-dimensional vectors. Following [*Rasskazov et al.*, 2018], we introduce an orthonormal basis of positive orientation in $\mathbb{R}^3$ consisting of unit vectors

$$\begin{aligned}\mathbf{l}&=(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta),\\ \mathbf{l}^{(1)}&=(\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ -\sin\theta),\\ \mathbf{l}^{(2)}&=(-\sin\varphi,\ \cos\varphi,\ 0),\end{aligned} \tag{33}$$

such that $\mathbf{q} = \varepsilon\mathbf{l}$. The solenoidality of $\langle \mathbf{V}'_0 \rangle$ and $\langle \mathbf{B}'_0 \rangle$ (see (29)) translates into the orthogonality $\mathbf{C_v} \cdot \mathbf{q} = \mathbf{C_b} \cdot \mathbf{q} = 0$, whereby

$$\mathbf{C_v} = C_1\mathbf{l}^{(1)} + C_2\mathbf{l}^{(2)}, \quad \mathbf{C_b} = C_3\mathbf{l}^{(1)} + C_4\mathbf{l}^{(2)}. \tag{34}$$

Scalar multiplying each equation (30) by $\mathbf{l}^{(1)}$ and $\mathbf{l}^{(2)}$ reduces the problem (30) to the eigenvalue problem for a $4 \times 4$ matrix

$$\mathrm{i}\mathbf{A}(\theta,\varphi) = \mathrm{i}\begin{bmatrix} \sum\limits_{k,j',j} A^{\mathbf{vv}}_{k,j'j} l^{(1)}_k l_{j'} l^{(1)}_j & \sum\limits_{k,j',j} A^{\mathbf{vv}}_{k,j'j} l^{(2)}_k l_{j'} l^{(1)}_j & \sum\limits_{k,j',j} A^{\mathbf{bv}}_{k,j'j} l^{(1)}_k l_{j'} l^{(1)}_j & \sum\limits_{k,j',j} A^{\mathbf{bv}}_{k,j'j} l^{(2)}_k l_{j'} l^{(1)}_j \\ \sum\limits_{k,j',j} A^{\mathbf{vv}}_{k,j'j} l^{(1)}_k l_{j'} l^{(2)}_j & \sum\limits_{k,j',j} A^{\mathbf{vv}}_{k,j'j} l^{(2)}_k l_{j'} l^{(2)}_j & \sum\limits_{k,j',j} A^{\mathbf{bv}}_{k,j'j} l^{(1)}_k l_{j'} l^{(2)}_j & \sum\limits_{k,j',j} A^{\mathbf{bv}}_{k,j'j} l^{(2)}_k l_{j'} l^{(2)}_j \\ -\sum\limits_{k,j} A^{\mathbf{vb}}_{k,j} l^{(1)}_k l^{(2)}_j & -\sum\limits_{k,j} A^{\mathbf{vb}}_{k,j} l^{(2)}_k l^{(2)}_j & -\sum\limits_{k,j} A^{\mathbf{bb}}_{k,j} l^{(1)}_k l^{(2)}_j & -\sum\limits_{k,j} A^{\mathbf{bb}}_{k,j} l^{(2)}_k l^{(2)}_j \\ \sum\limits_{k,j} A^{\mathbf{vb}}_{k,j} l^{(1)}_k l^{(1)}_j & \sum\limits_{k,j} A^{\mathbf{vb}}_{k,j} l^{(2)}_k l^{(1)}_j & \sum\limits_{k,j} A^{\mathbf{bb}}_{k,j} l^{(1)}_k l^{(1)}_j & \sum\limits_{k,j} A^{\mathbf{bb}}_{k,j} l^{(2)}_k l^{(1)}_j \end{bmatrix}. \tag{35}$$

We need to find the values of $\theta$ and $\varphi$ delivering the maximum growth rate $\gamma^{\mathbf{vb}}_\alpha$. Because of the factor i in front of the real matrix $\mathbf{A}$, this requires finding the eigenvalue of $\mathbf{A}$ with the largest in absolute value imaginary part.

In the case of the kinematic dynamo problem, $\mathbf{S}^{\mathbf{bb}}_k$ in (31.3) span the kernel of the magnetic induction operator $\mathscr{D} = \mathscr{D}_0$, only the vectors $\mathbf{A}^{\mathbf{bb}}_k$ are non-zero in (31), and the eigenvalue problem for the matrix (35) reduces to the eigenvalue problem for its $2 \times 2$ right lower corner cell. Its eigenvalues are [*Rasskazov et al.*, 2018]

$$\lambda^{\mathbf{b}}_{1_\pm}(\mathbf{l}) = -\frac{\mathrm{i}}{2}\sum_{k,j,m} \epsilon_{kjm} A^{\mathbf{bb}}_{k,j} l_m \pm \sqrt{a^{\mathbf{b}}}, \qquad a^{\mathbf{b}} = \mathbf{l} \cdot (\det {}^s\mathbf{A}^{\mathbf{bb}})\,({}^s\mathbf{A}^{\mathbf{bb}})^{-1}\mathbf{l},$$

where $\epsilon_{kjm}$ is the unit antisymmetric tensor and the entries of the symmetrized magnetic $\alpha$-effect tensor ${}^s\mathbf{A}^{\mathbf{bb}} = (\mathbf{A}^{\mathbf{bb}} + (\mathbf{A}^{\mathbf{bb}})^*)/2$ are ${}^sA^{\mathbf{bb}}_{k,j} = (A^{\mathbf{bb}}_{k,j} + A^{\mathbf{bb}}_{j,k})/2$. (The singularity, formally arising when the matrix ${}^s\mathbf{A}^{\mathbf{bb}}$ is non-invertible, is removed by expressing the inverse matrix in terms of the cofactors.) Hence the maximum growth rate due to the magnetic $\alpha$-effect is [*Rasskazov et al.*, 2018]

$$\gamma^{\mathbf{b}}_\alpha \equiv \max_{\theta,\varphi} \mathrm{Re}\,\lambda^{\mathbf{b}}_1(\theta,\varphi) = \sqrt{\max(\alpha_1\alpha_2,\, \alpha_2\alpha_3,\, \alpha_1\alpha_3)}, \tag{36}$$

where $\alpha_i$ are the three real eigenvalues of the $3 \times 3$ matrix ${}^s\mathbf{A}^{\mathbf{bb}}$. (For a symmetric magnetic $\alpha$-effect tensor $\mathbf{A}^{\mathbf{bb}}$, (36) stems from the results of [*Moffatt*, 1978], section 9.3 .)

In the case of the large-scale hydrodynamic stability problem, of the Navier–Stokes equation, $\mathscr{H} = \mathscr{H}_0$, and all the quantities (31) vanish except for the matrices $\mathbf{A}^{\mathbf{vv}}_k$ describing the AKA-effect. The eigenvalue problem for the matrix (35) reduces to the eigenvalue problem for the $2 \times 2$ cell in the upper left corner of (35). The two eigenvalues are

$$\lambda^{\mathbf{v}}_{1_\pm} = \frac{\mathrm{i}}{2}\sum_{k,j} a_{kj}(\delta^j_k - l_k l_j) \pm \frac{1}{2}\sqrt{a^{\mathbf{v}}},$$

where $\delta_k^j$ is the Kronecker symbol,

$$a_{kj} = \sum_{j'} A^{\mathbf{vv}}_{k,j'j} l_{j'},$$

$$a^{\mathbf{v}} = \Big(\sum_{k,j,m} \epsilon_{kjm} a_{kj} l_m\Big)^2 - \Big(\sum_{k,j} a_{kj} (l_k^{(1)} l_j^{(1)} - l_k^{(2)} l_j^{(2)})\Big)^2 - \Big(\sum_{k,j} a_{kj} (l_k^{(1)} l_j^{(2)} + l_k^{(2)} l_j^{(1)})\Big)^2$$
$$= \Big(\sum_{k,j,m} \epsilon_{kjm} a_{kj} l_m\Big)^2 - \Big(\operatorname{tr}{}^s a - \mathbf{l} \cdot {}^s a \mathbf{l}\Big)^2 + 4 \det{}^s a \, \mathbf{l} \cdot ({}^s a)^{-1} \mathbf{l}.$$

(The identity $\mathbf{l} = \mathbf{l}^{(1)} \times \mathbf{l}^{(2)}$ has been used to derive the expression for the discriminant $a^{\mathbf{v}}$.) Thus, the AKA-effect gives rise to large-scale modes of hydrodynamic linear perturbations that are growing or decaying only if the part of the AKA-effect tensor $A^{\mathbf{vv}}_{k,j'j}$, antisymmetric in $k$ and $j$, is non-zero, and then the frequency of oscillations in time is controlled by the symmetric part ${}^s a$ of the tensor $a$, where ${}^s a_{kj} = (a_{kj} + a_{jk})/2$. In contrast to the case of the kinematic dynamo problem, no analytical expression is available for the maximum growth rate $\gamma^{\mathbf{v}}_\alpha$, and it must be determined numerically (as well as in the case of the full $4\times 4$ matrix (35)).

### 5.2. *The combined MHD eddy diffusivity*

The operator of the combined MHD $\alpha$-effect in the l.h.s. of (30) can vanish identically, implying $\lambda_1 = 0$, $\langle P'_0 \rangle = 0$. The leading term coefficient in the expansion of the eigenvalue $\lambda$ is then $\lambda_2$, which is an eigenvalue of the MHD eddy diffusivity operator. We denote by $\gamma^{\mathbf{b}}_{\mathrm{e}}$, $\gamma^{\mathbf{v}}_{\mathrm{e}}$ and $\gamma^{\mathbf{bv}}_{\mathrm{e}}$ the maximum slow-time growth rates of the generated magnetic field and perturbations in the kinematic dynamo, hydrodynamic and MHD linear stability problems, respectively, due to the action of the eddy diffusivity and eddy viscosity. These values are defined as $\max_{\mathbf{q}} \operatorname{Re} \lambda_2$, where the coefficient $\lambda_2$ is the leading term in the expansion (26) of the eigenvalue $\lambda$.

This happens, for instance, if the MHD steady state $(\mathbf{V}(\mathbf{x}), \mathbf{B}(\mathbf{x}))$ is parity-invariant (see (23)). For such a steady state, the domain of the operator of linearization, $\mathscr{M}$, splits into two invariant subspaces: one is comprised of six-dimensional parity-invariant vector fields, the other one of parity-antiinvariant fields. Consequently, $\mathbf{S}^{\mathbf{vv}}_k, \mathbf{S}^{\mathbf{vb}}_k, \mathbf{S}^{\mathbf{bv}}_k, \mathbf{S}^{\mathbf{bb}}_k, \nabla S^{\mathbf{v}p}_k$ and $\nabla S^{\mathbf{b}p}_k$ are three-dimensional parity-antiinvariant fields, whereby all the quantities (31) vanish.

By linearity, solutions to the equation (27) for $n = 1$ take the form

$$(\mathbf{V}'_1, \mathbf{B}'_1, P'_1) = \sum_{k=1}^{3} \Big( \langle \mathbf{V}'_1 \rangle_k \mathbf{S}^{\mathbf{v}}_k(\mathbf{x}) + \langle \mathbf{B}'_1 \rangle_k \mathbf{S}^{\mathbf{b}}_k(\mathbf{x}) \Big) + \sum_{k=1}^{3} \sum_{m=1}^{3} \left( \mathbf{G}^{\mathbf{v}}_{mk} \frac{\partial \langle \mathbf{V}'_0 \rangle_k}{\partial X_m} + \mathbf{G}^{\mathbf{b}}_{mk} \frac{\partial \langle \mathbf{B}'_0 \rangle_k}{\partial X_m} \right). \quad (37)$$

Here vector fields $\mathbf{G}^{\mathbf{v}}_{mk}(\mathbf{x}) = (\mathbf{G}^{\mathbf{vv}}_{mk}, \mathbf{G}^{\mathbf{vb}}_{mk}, G^{\mathbf{v}p}_{mk})$ and $\mathbf{G}^{\mathbf{b}}_{mk}(\mathbf{x}) = (\mathbf{G}^{\mathbf{bv}}_{mk}, \mathbf{G}^{\mathbf{bb}}_{mk}, G^{\mathbf{b}p}_{mk})$ are $\mathbb{T}^3$-periodic zero-mean solutions to auxiliary problems:

$$\mathscr{M} \mathbf{G}^{\mathbf{v}}_{mk} = \Big( -2\nu \partial \mathbf{S}^{\mathbf{vv}}_k / \partial x_m + V_m \mathbf{S}^{\mathbf{vv}}_k - B_m \mathbf{S}^{\mathbf{vb}}_k + (S^{\mathbf{v}p}_k - \mathbf{V} \cdot \mathbf{S}^{\mathbf{vv}}_k + \mathbf{B} \cdot \mathbf{S}^{\mathbf{vb}}_k) \mathbf{e}_m,$$
$$-2\eta \partial \mathbf{S}^{\mathbf{vb}}_k / \partial x_m - \mathbf{V} S^{\mathbf{vb}}_{k,m} + V_m \mathbf{S}^{\mathbf{vb}}_k + \mathbf{B} S^{\mathbf{vv}}_{k,m} - B_m \mathbf{S}^{\mathbf{vv}}_k \Big), \quad (38.1)$$
$$\nabla_{\mathbf{x}} \cdot \mathbf{G}^{\mathbf{vv}}_{mk} = -S^{\mathbf{vv}}_{k,m}; \quad (38.2)$$

$$\mathscr{M} \mathbf{G}^{\mathbf{b}}_{mk} = \Big( -2\nu \partial \mathbf{S}^{\mathbf{bv}}_k / \partial x_m + V_m \mathbf{S}^{\mathbf{bv}}_k - B_m \mathbf{S}^{\mathbf{bb}}_k + (S^{\mathbf{b}p}_k - \mathbf{V} \cdot \mathbf{S}^{\mathbf{bv}}_k + \mathbf{B} \cdot \mathbf{S}^{\mathbf{bb}}_k) \mathbf{e}_m,$$
$$-2\eta \partial \mathbf{S}^{\mathbf{bb}}_k / \partial x_m - \mathbf{V} S^{\mathbf{bb}}_{k,m} + V_m \mathbf{S}^{\mathbf{bb}}_k + \mathbf{B} S^{\mathbf{bv}}_{k,m} - B_m \mathbf{S}^{\mathbf{bv}}_k \Big), \quad (39.1)$$
$$\nabla_{\mathbf{x}} \cdot \mathbf{G}^{\mathbf{bv}}_{mk} = -S^{\mathbf{bv}}_{k,m}. \quad (39.2)$$

The divergence of the magnetic component of (38.1) and (39.1) implies, respectively, relations

$$\nabla_{\mathbf{x}} \cdot \mathbf{G}^{\mathbf{vb}}_{mk} = -S^{\mathbf{vb}}_{k,m}, \qquad \nabla_{\mathbf{x}} \cdot \mathbf{G}^{\mathbf{bb}}_{mk} = -S^{\mathbf{bb}}_{k,m},$$

which guarantee together with (38.2) and (39.2) that the solenoidality conditions (29) hold true for $n = 1$. Vector fields $\mathbf{G}^{\mathbf{vv}}_{mk}, \mathbf{G}^{\mathbf{vb}}_{mk}, \mathbf{G}^{\mathbf{bv}}_{mk}, \mathbf{G}^{\mathbf{bb}}_{mk}, \nabla G^{\mathbf{v}p}_{mk}$ and $\nabla G^{\mathbf{b}p}_{mk}$ are parity-invariant.

The solvability condition for the equation (27) for $n = 2$, upon substituting (28) and (37), becomes

$$\nu\nabla^2_{\mathbf{X}}\langle \mathbf{V}'_0\rangle + \sum_{k,m,j}\left(\mathbf{D}^{\mathbf{vv}}_{jmk}\frac{\partial^2\left\langle \mathbf{V}'_0\right\rangle_k}{\partial X_m \partial X_j} + \mathbf{D}^{\mathbf{bv}}_{jmk}\frac{\partial^2\left\langle \mathbf{B}'_0\right\rangle_k}{\partial X_m \partial X_j}\right) - \nabla_{\mathbf{X}} P^*_1 = \lambda_2\langle \mathbf{V}'_0\rangle, \tag{40.1}$$

$$\eta\nabla^2_{\mathbf{X}}\langle \mathbf{B}'_0\rangle + \nabla_{\mathbf{X}} \times \sum_{k,m}\left(\mathbf{D}^{\mathbf{vb}}_{mk}\frac{\partial\left\langle \mathbf{V}'_0\right\rangle_k}{\partial X_m} + \mathbf{D}^{\mathbf{bb}}_{mk}\frac{\partial\left\langle \mathbf{B}'_0\right\rangle_k}{\partial X_m}\right) = \lambda_2\langle \mathbf{B}'_0\rangle. \tag{40.2}$$

We have denoted

$$\mathbf{D}^{\mathbf{vv}}_{jmk} = \left\langle -V_j\mathbf{G}^{\mathbf{vv}}_{mk} - \mathbf{V}G^{\mathbf{vv}}_{mk,j} + B_j\mathbf{G}^{\mathbf{vb}}_{mk} + \mathbf{B}G^{\mathbf{vb}}_{mk,j}\right\rangle, \tag{41.1}$$

$$\mathbf{D}^{\mathbf{bv}}_{jmk} = \left\langle -V_j\mathbf{G}^{\mathbf{bv}}_{mk} - \mathbf{V}G^{\mathbf{bv}}_{mk,j} + B_j\mathbf{G}^{\mathbf{bb}}_{mk} + \mathbf{B}G^{\mathbf{bb}}_{mk,j}\right\rangle, \tag{41.2}$$

$$\mathbf{D}^{\mathbf{vb}}_{mk} = \left\langle \mathbf{V} \times \mathbf{G}^{\mathbf{vb}}_{mk} - \mathbf{B} \times \mathbf{G}^{\mathbf{vv}}_{mk}\right\rangle, \tag{41.3}$$

$$\mathbf{D}^{\mathbf{bb}}_{mk} = \left\langle \mathbf{V} \times \mathbf{G}^{\mathbf{bb}}_{mk} - \mathbf{B} \times \mathbf{G}^{\mathbf{bv}}_{mk}\right\rangle. \tag{41.4}$$

The second-order partial differential operator in the l.h.s. of (40) is called the *operator of the combined MHD eddy diffusion*; in general, it is anisotropic.

As in the case of the combined $\alpha$-effect operator, the problem (40) admits mean-field eigenfunctions (32) satisfying (34). Their coefficients $(C_1, C_2, C_3, C_4)$ are eigenvectors of the $4 \times 4$ matrix

$$\mathbf{D}(\theta,\varphi) = -\begin{bmatrix} \nu + \sum\limits_{k,j} d^{\mathbf{vv}}_{kj} l^{(1)}_k l^{(1)}_j & \sum\limits_{k,j} d^{\mathbf{vv}}_{kj} l^{(2)}_k l^{(1)}_j & \sum\limits_{k,j} d^{\mathbf{bv}}_{kj} l^{(1)}_k l^{(1)}_j & \sum\limits_{k,j} d^{\mathbf{bv}}_{kj} l^{(2)}_k l^{(1)}_j \\ \sum\limits_{k,j} d^{\mathbf{vv}}_{kj} l^{(1)}_k l^{(2)}_j & \nu + \sum\limits_{k,j} d^{\mathbf{vv}}_{kj} l^{(2)}_k l^{(2)}_j & \sum\limits_{k,j} d^{\mathbf{bv}}_{kj} l^{(1)}_k l^{(2)}_j & \sum\limits_{k,j} d^{\mathbf{bv}}_{kj} l^{(2)}_k l^{(2)}_j \\ -\sum\limits_{k,j} d^{\mathbf{vb}}_{kj} l^{(1)}_k l^{(2)}_j & -\sum\limits_{k,j} d^{\mathbf{vb}}_{kj} l^{(2)}_k l^{(2)}_j & \eta - \sum\limits_{k,j} d^{\mathbf{bb}}_{kj} l^{(1)}_k l^{(2)}_j & -\sum\limits_{k,j} d^{\mathbf{bb}}_{kj} l^{(2)}_k l^{(2)}_j \\ \sum\limits_{k,j} d^{\mathbf{vb}}_{kj} l^{(1)}_k l^{(1)}_j & \sum\limits_{k,j} d^{\mathbf{vb}}_{kj} l^{(2)}_k l^{(1)}_j & \sum\limits_{k,j} d^{\mathbf{bb}}_{kj} l^{(1)}_k l^{(1)}_j & \eta + \sum\limits_{k,j} d^{\mathbf{bb}}_{kj} l^{(2)}_k l^{(1)}_j \end{bmatrix} \tag{42}$$

obtained by scalar multiplying the flow and magnetic components of (40) by $\mathbf{l}^{(1)}$ and $\mathbf{l}^{(2)}$ (33). Here

$$d^{\mathbf{vv}}_{kj} = \sum_{j',m} D^{\mathbf{vv}}_{j'mk,j} l_{j'} l_m, \quad d^{\mathbf{bv}}_{kj} = \sum_{j',m} D^{\mathbf{bv}}_{j'mk,j} l_{j'} l_m, \quad d^{\mathbf{vb}}_{kj} = \sum_{m} D^{\mathbf{vb}}_{mk,j} l_m, \quad d^{\mathbf{bb}}_{kj} = \sum_{m} D^{\mathbf{bb}}_{mk,j} l_m.$$

In the large-scale kinematic dynamo problem, only the vectors $\mathbf{D}^{\mathbf{bb}}_{mk}$ are non-zero, and in the large-scale hydrodynamic stability problem, only $\mathbf{D}^{\mathbf{vv}}_{jmk}$ do not vanish in (41). This leaves us with the eigenvalue problems for the $2 \times 2$ right lower or left upper corner cells of (42), respectively.

The eigenvalue equation for the kinematic dynamo problem takes the form

$$(\lambda_2^{\mathbf{b}}+\eta)^2-\operatorname{tr}\mathbf{w}(\lambda_2^{\mathbf{b}}+\eta)+\det\mathbf{w}=0\quad\text{for}\quad \mathbf{w}=\begin{bmatrix}\sum\limits_{k,j} d_{kj}^{\mathbf{bb}}l_k^{(1)}l_j^{(2)} & \sum\limits_{k,j} d_{kj}^{\mathbf{bb}}l_k^{(2)}l_j^{(2)}\\ -\sum\limits_{k,j} d_{kj}^{\mathbf{bb}}l_k^{(1)}l_j^{(1)} & -\sum\limits_{k,j} d_{kj}^{\mathbf{bb}}l_k^{(2)}l_j^{(1)}\end{bmatrix}.$$

We can reduce the degree of the trace of $\mathbf{w}$ (regarded as a polynomial in components of the vectors (33)), from 3 to 2, and of the determinant from 6 to 4:

$$\operatorname{tr}\mathbf{w}=\sum_{k,j,m}\epsilon_{kjm}d_{kj}^{\mathbf{bb}}l_m,\qquad \det\mathbf{w}=\sum_{k,k',j,j',m,m'}\frac{\epsilon_{kk'm}\epsilon_{jj'm'}}{2}d_{kj}^{\mathbf{bb}}d_{k'j'}^{\mathbf{bb}}l_m l_{m'}.$$

If the respective eigenvalue is real, then the maximum growth rate due to magnetic eddy diffusivity in terms of the optimal values $\theta_{\text{opt}}$ and $\varphi_{\text{opt}}$ is

$$\gamma_{\mathrm{e}}^{\mathbf{b}}\equiv\max_{\theta,\varphi}\operatorname{Re}\lambda_2^{\mathbf{b}}(\theta,\varphi)=\left.\frac{\partial\det\mathbf{w}/\partial\theta}{\partial\operatorname{tr}\mathbf{w}/\partial\theta}\right|_{\theta=\theta_{\text{opt}},\,\varphi=\varphi_{\text{opt}}}-\eta=\left.\frac{\partial\det\mathbf{w}/\partial\varphi}{\partial\operatorname{tr}\mathbf{w}/\partial\varphi}\right|_{\theta=\theta_{\text{opt}},\,\varphi=\varphi_{\text{opt}}}-\eta$$

(provided the denominators do not vanish).

The similarity of the structure of the matrices (35) and (42) and a simple algebra also yield

$$\lambda_{2_\pm}^{\mathbf{b}}(\mathbf{l})=\frac{1}{2}\sum_{k,j,m}\epsilon_{kjm}d_{kj}^{\mathbf{bb}}l_m\pm\sqrt{d^{\mathbf{b}}}-\eta,$$

$$d^{\mathbf{b}}=-\mathbf{l}\cdot(\det{}^s\mathbf{d}^{\mathbf{bb}})({}^s\mathbf{d}^{\mathbf{bb}})^{-1}\mathbf{l}=-\sum_{k,k',j,j',m,m'}\frac{\epsilon_{kk'm}\epsilon_{jj'm'}}{2}\,{}^sd_{kj}^{\mathbf{bb}}\,{}^sd_{k'j'}^{\mathbf{bb}}l_m l_{m'},$$

where the symmetrized 3×3 matrix ${}^s\mathbf{d}^{\mathbf{bb}}=(\mathbf{d}^{\mathbf{bb}}+(\mathbf{d}^{\mathbf{bb}})^*)/2$ has the entries ${}^sd_{kj}^{\mathbf{bb}}=(d_{kj}^{\mathbf{bb}}+d_{jk}^{\mathbf{bb}})/2$ (no singularity arises when the matrix ${}^s\mathbf{d}^{\mathbf{bb}}$ is non-invertible).

Employing again the similarity of the structure of the matrices (35) and (42), for the eddy viscosity acting on large-scale amagnetic perturbations of the flow, we find

$$\lambda_{2_\pm}^{\mathbf{v}}=-\frac{1}{2}\sum_{k,j}d_{kj}^{\mathbf{vv}}(\delta_k^j-l_kl_j)\pm\sqrt{d^{\mathbf{v}}}-\nu,$$

$$d^{\mathbf{v}}=-\Big(\sum_{k,j,m}\epsilon_{kjm}d_{kj}^{\mathbf{vv}}l_m\Big)^2+\Big(\sum_{k,j}d_{kj}^{\mathbf{vv}}(l_k^{(1)}l_j^{(1)}-l_k^{(2)}l_j^{(2)})\Big)^2+\Big(\sum_{k,j}d_{kj}^{\mathbf{vv}}(l_k^{(1)}l_j^{(2)}+l_k^{(2)}l_j^{(1)})\Big)^2.$$

Thus, the growth of large-scale modes of hydrodynamic linear perturbations is possible only if the eddy viscosity tensor $D_{j'mk,j}^{\mathbf{vv}}$ features a non-zero part, symmetric in $k$ and $j$.

In all the three cases, maximization of the growth rates in $\theta,\varphi$ must be implemented numerically.

### 5.3. *Computation of the combined MHD eddy diffusivity tensor*

It is known (see, e.g., [*Andrievsky et al.*, 2015; *Rasskazov et al.*, 2018]) that, in the large-scale kinematic dynamo problem, computing the tensor of magnetic eddy diffusivity can be significantly accelerated: instead of solving nine problems (39), we solve three auxiliary problems for the adjoint operator, $\mathscr{D}^*\mathbf{Z}_n^{\mathbf{b}}=\mathbf{V}\times\mathbf{e}_n$, and use the equivalent expression

$$D_{mk,n}^{\mathbf{bb}}=\langle\langle\mathbf{Z}_n^{\mathbf{b}},\,2\eta\,\partial\mathbf{S}_k^{\mathbf{bb}}/\partial x_m+\mathbf{e}_m\times(\mathbf{V}\times\mathbf{S}_k^{\mathbf{bb}})\rangle\rangle.$$

(The numerical load of finding one field $\mathbf{G}_{mk}^{\mathbf{bb}}$ is approximately the same as that of finding one field $\mathbf{Z}_n^{\mathbf{b}}$.) When computing the combined MHD eddy diffusivity tensor, the same approach is applicable: while using the formulae (41) requires solving nine problems (38) and nine problems (39), it suffices to compute $\mathbb{T}^3$-periodic parity-invariant solutions $\mathbf{Z}^{\mathbf{v}}_{nj}=(\mathbf{Z}^{\mathbf{vv}}_{nj},\mathbf{Z}^{\mathbf{vb}}_{nj})$ and $\mathbf{Z}^{\mathbf{b}}_{n}=(\mathbf{Z}^{\mathbf{bv}}_{n},\mathbf{Z}^{\mathbf{bb}}_{n})$ to six auxiliary problems for the adjoint operator of the form

$$\mathscr{M}^*\mathbf{Z}^{\mathbf{v}}_{nj}=(V_j\mathbf{e}_n+V_n\mathbf{e}_j,-B_j\mathbf{e}_n-B_n\mathbf{e}_j)$$

and three problems of the form

$$\mathscr{M}^*\mathbf{Z}^{\mathbf{b}}_{n}=(\mathbf{e}_n\times\mathbf{B},-\mathbf{e}_n\times\mathbf{V}).$$

Then

$$\begin{aligned}D^{\xi\mathbf{v}}_{jmk,n}&=-\langle\langle\mathscr{M}^*\mathbf{Z}^{\mathbf{v}}_{nj},\mathbf{G}^{\xi}_{mk}\rangle\rangle\\&=\langle\langle\mathbf{Z}^{\mathbf{vv}}_{nj},\ 2\nu\,\partial\mathbf{S}^{\xi\mathbf{v}}_k/\partial x_m-V_m\mathbf{S}^{\xi\mathbf{v}}_k+B_m\mathbf{S}^{\xi\mathbf{b}}_k-(S^{\xi p}_k-\mathbf{V}\cdot\mathbf{S}^{\xi\mathbf{v}}_k+\mathbf{B}\cdot\mathbf{S}^{\xi\mathbf{b}}_k)\mathbf{e}_m\rangle\rangle\\&\quad+\langle\langle\mathbf{Z}^{\mathbf{vb}}_{nj},\ 2\eta\,\partial\mathbf{S}^{\xi\mathbf{b}}_k/\partial x_m+\mathbf{V}S^{\xi\mathbf{b}}_{k,m}-V_m\mathbf{S}^{\xi\mathbf{b}}_k-\mathbf{B}S^{\xi\mathbf{v}}_{k,m}+B_m\mathbf{S}^{\xi\mathbf{v}}_k\rangle\rangle,\\D^{\xi\mathbf{b}}_{mk,n}&=-\langle\langle\mathscr{M}^*\mathbf{Z}^{\mathbf{b}}_{n},\mathbf{G}^{\xi}_{mk}\rangle\rangle\\&=\langle\langle\mathbf{Z}^{\mathbf{bv}}_{n},\ 2\nu\,\partial\mathbf{S}^{\xi\mathbf{v}}_k/\partial x_m-V_m\mathbf{S}^{\xi\mathbf{v}}_k+B_m\mathbf{S}^{\xi\mathbf{b}}_k-(S^{\xi p}_k-\mathbf{V}\cdot\mathbf{S}^{\xi\mathbf{v}}_k+\mathbf{B}\cdot\mathbf{S}^{\xi\mathbf{b}}_k)\mathbf{e}_m\rangle\rangle\\&\quad+\langle\langle\mathbf{Z}^{\mathbf{bb}}_{n},\ 2\eta\,\partial\mathbf{S}^{\xi\mathbf{b}}_k/\partial x_m+\mathbf{V}S^{\xi\mathbf{b}}_{k,m}-V_m\mathbf{S}^{\xi\mathbf{b}}_k-\mathbf{B}S^{\xi\mathbf{v}}_{k,m}+B_m\mathbf{S}^{\xi\mathbf{v}}_k\rangle\rangle,\end{aligned}$$

where the metaindex $\xi$ replaces indices $\mathbf{v}$ or $\mathbf{b}$. The gain in efficiency is smaller when computing the eddy viscosity tensor $\mathbf{D}^{\mathbf{vv}}_{mk}$ in the large-scale hydrodynamic stability problem: solving nine auxiliary problems (38) for $\mathbf{G}^{\mathbf{vv}}_{mk}$ is replaced by solving six auxiliary problems for the adjoint operator $\mathscr{H}^*\mathbf{Z}^{\mathbf{v}}_{nj}=V_j\mathbf{e}_n+V_n\mathbf{e}_j$ and applying

$$D^{\mathbf{vv}}_{jmk,n}=-\langle\langle\mathscr{H}^*\mathbf{Z}^{\mathbf{v}}_{nj},\mathbf{G}^{\mathbf{vv}}_{mk}\rangle\rangle=\langle\langle\mathbf{Z}^{\mathbf{v}}_{nj},\,2\nu\,\partial\mathbf{S}^{\mathbf{vv}}_k/\partial x_m-V_m\mathbf{S}^{\mathbf{vv}}_k-(S^{\mathbf{v}p}_k-\mathbf{V}\cdot\mathbf{S}^{\mathbf{vv}}_k)\mathbf{e}_m\rangle\rangle.$$

## 6. Concluding remarks

We have examined the mathematical and computational aspects of the analysis of the linear stability of steady space-periodic flows and MHD states to Bloch eigenmodes. Three linear stability problems have been considered: the kinematic dynamo problem, the hydrodynamic and MHD stability problem. They all reduce to eigenvalue problems for the modified operators of linearization $\mathscr{D}_\mathbf{q}$ (19.1), $\mathscr{H}_\mathbf{q}$ (18.1) and $\mathscr{M}_\mathbf{q}$ (7), respectively, that are solved in the periodicity box of the steady state that is perturbed. This is computationally advantageous, since the need to simultaneously resolve numerically multiple spatial scales is thus avoided.

Application of an algorithm of the steepest descent type for computing the dominant growth rate requires evaluating its gradient at each step. We have derived expressions for $\partial\gamma/\partial q_m$ in terms of the dominant eigenfunctions of the adjoint operators $\mathscr{D}^*_\mathbf{q}$ (19.2), $\mathscr{H}^*_\mathbf{q}$ (18.2) and $\mathscr{M}^*_\mathbf{q}$ (11) (see (14) for the MHD stability problem; for the kinematic dynamo and hydrodynamic stability problems, the expressions are obtained from (14) by setting $\mathbf{B}=\mathbf{v}=0$ or $\mathbf{B}=\mathbf{b}=0$). They are suitable for computing the gradient and have been employed for proving that half-integer wave vectors $\mathbf{q}$ are stationary points of $\gamma$, when a parity-invariant space-periodic flow or MHD steady state is perturbed, or when the eigenvalue of the operator of linearization is real. We have also discussed computation of the Hessian of $\gamma(\mathbf{q})$, which is needed for numerical verification that a maximum $\gamma$ is obtained.

It was demonstrated within the framework of the multiscale stability theory that when the spatial scale separation is high, the $\alpha$-effect or, for parity-invariant steady states, the eddy diffusivity govern the evolution of the large-scale perturbations. We have overviewed these results from the viewpoint of the prospective numerical application. The $\alpha$-effect and eddy diffusivity tensors are expressed in the terms of solutions to the auxiliary problems. Again, our goal is to compute the maximum, over the direction of the infinitesimal (as required in this theory) wave vector **q**, slow-time growth rates of the modes arising due to the action of these phenomena. In the case of the magnetic $\alpha$-effect, the maximum growth rate can be readily determined (36) from the eigenvalues of the symmetrized $\alpha$-effect tensor. No similar direct relations between the dominant growth rates and the $\alpha$-effect or eddy diffusivity tensors are known in any other case (i.e., in the presence of the $\alpha$-effect in the hydrodynamic and MHD stability problems, or in the presence of the eddy diffusivity in any of the three stability problems). It is desirable to establish such relations; the well-known analogy between the evolution of the magnetic field and fluid vorticity suggests that at least for the AKA-effect this may be possible.

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