Pis'ma v ZhETF, vol. 119, iss. 5, pp. 388 – 389

Scattering of linear waves on a soliton[1)]

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Submitted 27 December 2023
Resubmitted 5 February 2024
Accepted 6 February 2024

DOI: 10.31857/S1234567824050100, EDN: GWRTEW

Introduction. Solitons are stable bound states which exist on a classical as well as quantum mechanical levels in nonlinear field theories. They have a finite stable shape in space, which preserves over time and during free propagation. The dispersion in the solitons is balanced by focusing nonlinear effects. Solitons are studied in many branches of physics such as optics [1], plasma physics [2], condensed matter physics [3], cosmology [4] and other natural sciences.

In this paper, we consider the scattering of linear high-frequency waves on non-topological solitons in the model of a nonlinear non-integrable Schrödinger equation. We show that, in contrast to the integrable case in which the incoming wave causes a shift in the soliton's position [5], the presence of non-integrable nonlinear interaction leads to a non-trivial scattering picture. Nonlinear interactions give rise to waves of double, triple, and so on, frequencies, while the soliton grows by absorbing particles from the incoming wave. We propose an analytical method for describing the scattering pattern of high- frequency waves on a soliton. To validate the method, we compare its predictions with the results of numerical simulations and observe a strong agreement.

Setup. We start by considering the nonlinear Schrödinger equation in dimensionless variables

$$i\partial_t\psi = -\frac{1}{2}\partial_x^2\psi + V(|\psi|^2)\psi \ , \tag{1}$$

where the last term introduces nonlinearity. For numerical simulation, we have chosen the potential

$$V(|\psi|^2) = -\lambda|\psi|^2 + g|\psi|^4 \ . \tag{2}$$

Evolution governed by Eq. (1) conserves a number of quantities: the particle number (norm) N, $N = \int dx\, |\psi|^2$, the energy of the system E, $E = \int dx \left[\frac{1}{2}|\partial_x\psi|^2 + \int\limits_0^{|\psi|^2} ds\, V(s)\right]$.

Suppose that Eq. (1) admits a soliton solution of the form

$$\psi_s(t,x) = f(x)\mathrm{e}^{-i\gamma t}, \tag{3}$$

where real function $f(x)$ gives the soliton profile. It satisfies $f(-x) = f(x)$ and $f(x) \to 0$ for $|x| \to \infty$. The soliton frequency is negative, $\gamma < 0$. It determines the binding energy of nonrelativistic particles in a soliton according to the relation

$$dE = \gamma dN, \tag{4}$$

which directly follows from Eqs. (1), (3). The solution is supposed to be classically stable in according to the Vakhitov–Kolokolov stability criterion [6].

Below we will consider a wave packet moving from large negative x to the right and scattering on a soliton (3) centered at $x = 0$,

$$\psi_0 = A(t,x)\mathrm{e}^{-i\omega t + ipx + i\phi_0} \ , \qquad \omega = \frac{p^2}{2} \ , \tag{5}$$

where A specifies a wave packet shape and ϕ_0 is a constant phase. In what follows, we will assume that $|V(A^2)| \ll \omega$, so the wave packet freely propagates to the region of interaction with the soliton. In addition, we will consider wave packets with a width σ much larger than the wavelength, $\sigma \gg 2\pi/p$ so that the change in its shape during movement can be neglected. We solve numerically the nonlinear Schrödinger equation (1) with the potential (2) starting at $t = 0$ with the initial wave function $\psi = \psi_0 + \psi_s$. Here ψ_s is the wave function of the soliton centered at $x = 0$ and ψ_0 is an incident Gaussian wave packet, Eq. (5) with

$$A(0,x) = A\,\mathrm{e}^{-(x-x_0)^2/2\sigma^2} \ , \tag{6}$$

localized at large negative $x \simeq x_0$ far away from the soliton, where $A = 0.01$, $x_0 = -1500$, $\sigma = 100$ and $p = 1$. The results are demonstrated in the movie [7].

[1)]Supplementary materials are available for this article at DOI: 10.1134/S0021364024600319 and are accessible for authorized users.

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The incident wave packet strikes the soliton and then separates into reflected and transmitted parts. Each part consists of several wave packets moving with different velocities. In particular, the wave packets with maximal amplitudes, transmitted $\psi_1^{(tr)}$ and reflected $\psi_1^{(re)}$, move at the velocity of the incident wave packet $v_1 = \sqrt{2\omega}$. They are the result of quantum-mechanical-like scattering on a potential well produced by a soliton. We have numerically verified that for $A \lesssim 0.01$ the amplitude of $\psi_1^{(re)}$ scales as A, while the amplitudes of ψ_2 and ψ_3, which appear due to nonlinear interaction of an incident wave packet and a soliton, are proportional to A^2 and A^3 respectively. This fact will be used to construct a perturbation theory with respect to A.

The above argument indicates that the scattering of the wave packet is accompanied by an increase in the number of particles in the soliton. This is confirmed by numerical simulations. The change in the norm of the soliton as a function of the incident wave packet frequency is presented in Fig. 1 by solid line. Surprisingly that this change coincides with high accuracy with the number of particles in the transmitted wave packet $\psi_2^{(tr)}$ (dahsed line in Fig. 1),

$$\Delta N_s = N_2^{(tr)}. \tag{7}$$

Considering that $\psi_2^{(tr)} \propto A^2$ the number of particles captured by the soliton is parametrically small, $\Delta N_s \propto A^4$.

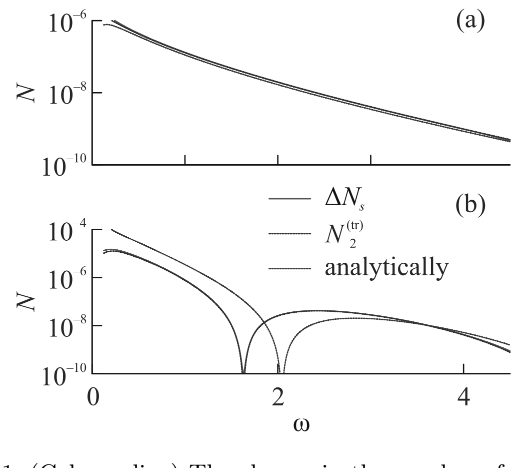

Fig. 1. (Color online) The change in the number of particles of the soliton ΔN_s (red solid lines) with $\gamma = -0.125$ (a) and with $\gamma = -0.1874375$ (b) and the number of particles in the $\psi_2^{(tr)}$ wave packet $N_2^{(tr)}$ (blue dashed lines) as a function of the incident wave packet frequency. The green dotted lines represent the estimation for the norm $\psi_2^{(tr)}$, which was calculated using our analytical method based on the Born approximation

Results. In this study, we have demonstrated that wave scattering by a soliton is accompanied by the generation of waves with frequencies that are multiples of the incident wave frequency, minus the soliton frequency. Additionally, we have observed that the soliton undergoes growth by absorbing particles from the incident wave into its ground state, while the exited states remain unoccupied. This fact can be understood by adopting the methods of quantum field theory[3)]. The nonlinear self-interaction term $|\psi|^4$ is responsible for scattering of two particles from the incident wave packet into two other states. The occupation number of the soliton ground state is large, thus the transition to this state is strongly Bose-enhanced. Based on the Born approximation, a wave packet with a frequency of $2\omega - \gamma$ is also produced in this process. According to the conservation laws of total energy and total number of particles, it follows that the norm of this wave packet should be equal to the number of particles absorbed by the soliton.

We thank M. Smolyakov and E. Nugaev for numerous discussions and interest in the work.

Funding. This work was supported by the grant Russian Science Foundation 22-12-00215 and, in its numerical part, by the “BASIS” foundation. Numerical calculations were performed on the Computational Cluster of the Theoretical Division of Institute for Nuclear Research of the Russian Academy of Sciences.

Conflict of interest. Authors declare that there is no conflict of interest.

This is an excerpt of the article “Scattering of linear waves on a soliton”. Full text of the paper is published in JETP Letters journal. DOI: 10.1134/S0021364024600319

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[3)] It is important to emphasize that quantum physics is not essential for the soliton-wave interaction.